OVERLAPPING DOMAIN DECOMPOSITION METHODS FOR TOTAL VARIATION DENOISING[*]

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Abstract. Alternating and parallel overlapping domain decomposition methods for the minimization of the total variation are presented. Their derivation is based on the predual formulation of the total variation minimization problem. In particular, the predual total variation minimization problem is decomposed into overlapping domains yielding subdomain problems in the respective dual space. Subsequently these subdomain problems are again dualized, forming a splitting algorithm for the original total variation minimization problem. The convergence of the proposed domain decomposition methods to a solution of the global problem is proven. In contrast to other works, the analysis is carried out in an infinite dimensional setting. Numerical experiments are shown to support the theoretical results and to demonstrate the effectiveness of the algorithms.

Key words. domain decomposition, total variation minimization, convex optimization, image restoration, convergence analysis, subspace correction, locally weighted total variation

AMS subject classifications. 68U10, 94A08, 49M27, 65K10, 90C06

1. Introduction. Minimizing the total variation in the context of image denoising was first proposed in [49] and has gained a lot of attention since then, because it allows to preserve edges and discontinuities in images. In this approach one typically minimizes an energy consisting of a data-fidelity term, which enforces the consistency between the observed and obtained image, a total variation term, as regularizer, and a parameter weighting the importance of these two terms. The choice of the data-term depends on the type of noise contamination. Here we assume that the observed image g is corrupted by additive Gaussian noise, i.e., $g = \hat{u} + \eta$, where $\hat{u}$ is the unknown true image and η represents the noise. For such images usually a quadratic L^2-data fidelity term is chosen; see for example [6, 9] and references therein. That is, the image $\hat{u}$ is recovered from the observation $g \in L^2(\Omega)$ by solving

$$(1.1) \qquad \underset{u \in L^2(\Omega)}{\arg\min} \left\{ J(u) := \frac{1}{2} \|u - g\|_{L^2(\Omega)}^2 + \alpha \int_\Omega |Du| \right\}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary, $\alpha > 0$ is the regularization parameter and $\int_\Omega |Du| = \sup\{\int_\Omega u \operatorname{div} \mathbf{p} \, dx \ : \ \mathbf{p} \in C_0^1(\Omega, \mathbb{R}^2), |\mathbf{p}|_{\ell^2} \le 1$ almost everywhere (a.e.) in $\Omega\}$ denotes the total variation of u in Ω [3, 29] with $C_0^1(\Omega, \mathbb{R}^2)$ being the space of continuously differentiable vector valued functions with compact support in Ω. Here and in the rest of the paper, bold letters indicate vector valued functions. Note that different fidelity terms have been considered in connection with other types of noise, as impulse noise [2, 46, 47], Poisson noise [44], multiplicative noise [4], Rician noise [28], mixed noise [7, 15, 19, 32, 40, 42].

Existing state-of-the-art methods for solving (1.1), as described in [6, 9], perform well for small- and medium-scale problems. However, they are not able to perform in realistic CPU-time large-scale problems. Such large-scale problems, occur nowadays in nearly every application in image reconstruction, due to the improvement of hardware. In order to deal with these huge problems, new methods need to be developed.

It has been shown multiple times [48, 53], that domain decomposition methods are one of the most successful methods to construct efficient solvers for large-scale problems. This is due to the fact that they allow for decomposing the original problem into a sequence of smaller problems, which may be distributed on several processors with the possibility of parallelization. While for

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domain decomposition methods minimizing smooth energies, the convergence, rate of convergence, and independence of the rate of convergence from the mesh size of discretization are well established, not much is known for minimizing nonsmooth and nonadditive functionals. Note, that the energy in (1.1) is nonsmooth and nonadditive, due to the presence of the total variation. We remark, that for nonsmooth problems, the resulting splitting algorithms still work fine as long as the energy splits additively with respect to the domain decomposition. For such problems convergence and sometimes even rate of convergence are ensured; see for example [23, 54]. Moreover, for image deblurring problems preconditioning effects of a specific subspace correction algorithm for minimizing a nonsmooth energy are shown in [55]. For nonsmooth and nonadditive energies, however, the research on subspace correction methods is far from being complete, and for some problem classes counterexamples do exist indicating failure of splitting techniques; see e.g. [24, 57].

For introducing domain decomposition strategies for solving problem (1.1) the major difficulty lies in the correct treatment of the interfaces of the domain decomposition patches, i.e. the preservation of crossing discontinuities and the correct matching where the solution is continuous. We emphasize that for well-known approaches as those in [8, 11, 51, 52] it is not clear yet whether they indeed converge to a global minimizer for nonsmooth and nonadditive problems, as any convergence theory in this direction is missing. Nevertheless, in [14] and [58] the subspace correction approaches of [51, 52] are used to solve smoothed versions of (1.1).

In [25, 26, 27, 43, 50] nonoverlapping and overlapping domain decomposition methods for total variation minimization are described. Thereby, the convex objective under some linear constraint, ensuring the correct treatment of the internal interfaces, is iteratively minimized on each subdomain. While in these papers an implementation guaranteeing convergence and monotonic decay of the objective energy is provided, convergence to the global minimizer of the total variation problem cannot be ensured, in general. In [26] a proof establishing convergence of overlapping domain decomposition algorithms to the global solution in a discrete setting is presented, which, however, only holds for one-dimensional problems. It is not clear yet how to extend this proof to any finite dimensional space without introducing additional assumptions. Moreover, an extension to infinite dimensional spaces is also missing till now.

For a class of nonsmooth and nonadditive convex variational problems with a combined L^1/L^2 data fidelity term in [32, 33] overlapping and nonoverlapping domain decomposition methods are presented. In particular, their convergence and monotonic decay of the energy is theoretically ensured. Moreover, an estimate of the distance of the limit point obtained from the domain decomposition methods to the true global minimizer is derived. With the help of this estimate it is demonstrated by numerical experiments that the domain decomposition methods indeed generate sequences which converge to the global minimizer. However, a theoretical proof of convergence of the domain decomposition methods to the global solution is missing.

Without any rigorous theoretical analysis in [21] domain decomposition methods for solving (1.1) by graph cuts are introduced and applied to the task of image segmentation. Moreover, for image segmentation using the Chan-Vese model [12] and based on a primal-dual formulation recently nonoverlapping domain decomposition methods are presented in [20].

In order to tackle the difficulties due to the minimization of a nonsmooth and nonadditive objective in (1.1), in [13, 34] a predual problem of (1.1), see [31, 38] for the derivation of the latter, is considered. In fact, a predual of (1.1) reads:

$$(1.2) \qquad \min \frac{1}{2} || \operatorname{div} \mathbf{p} + g ||^2_{L^2(\Omega)} \ \text{ over } \mathbf{p} \in H_0(\operatorname{div}, \Omega)$$

$$\text{s.t. } |\mathbf{p}(x)|_{\ell^2} \le \alpha \text{ for almost all (f.a.a.) } x \in \Omega,$$

where $H_0(\operatorname{div}, \Omega) := \{\mathbf{v} \in \mathbb{L}^2(\Omega) : \operatorname{div} \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ with $\mathbb{L}^2(\Omega) := L^2(\Omega) \times L^2(\Omega)$ and $\mathbf{n}$ being the outward unit normal on $\partial\Omega$. Note, that the solution u^* of (1.1) and a solution $\mathbf{p}^*$ of (1.2) are related by

$$(1.3) \qquad\qquad\qquad\qquad u^* = \operatorname{div} \mathbf{p}^* + g,$$

see [31]. The smooth objective and the box-constraint seem more amenable to a domain decomposition than the structure of (1.1). In fact, in [34] nonoverlapping domain decomposition methods

for the problem in (1.2), where instead of $|\mathbf{p}(x)|_{\ell^2} \leq \alpha$ the constraint $|\mathbf{p}(x)|_{\ell^\infty} \leq \alpha$ is considered, are introduced and shown to converge to a minimizer of the global problem in a discrete setting. It is still an open problem to show such a convergence result in an infinite dimensional setting.

Based on the nonoverlapping domain decomposition strategy in [34] for the predual problem (1.2) in [45] a nonoverlapping algorithm for a discretized version of the primal problem (1.1) is constructed. Thereby the following strategy is pursued: The domain decomposition method in [34] is constituted by its subdomain problems. Then the dual problems of these subdomain problems are computed, yielding a sequence of nonoverlapping subdomain problems of the primal problem. Due to the predualization and dualization, the finally constituted domain decomposition method of the discretized primal problem looks different than the splitting strategies presented in [27, 32]. Using the connection between the primal subdomain problems and their predual counterparts allows in [45] to show analytically the convergence of the nonoverlapping domain deomposition methods to the minimizer of the global problem in a discrete setting.

While convergence to a solution of the global problem (1.1) or (1.2) for nonoverlapping domain decomposition methods in an infinite dimensional setting has not yet been theoretically proven, there exist overlapping domain decomposition methods for the predual problem (1.2) with such desirable convergence properties. In particular, in [13] overlapping domain decomposition methods for the predual problem (1.2) are introduced and the convergence to the true minimizer of the global problem is shown analytically in an infinite dimensional setting, together with a convergence rate. We note, that this results cannot be (directly) extended to the nonoverlapping case. This is due to the fact, that the overlapping decomposition in [13] is determined by a partition of unity function, whose partitions have to be sufficiently smooth. This essential smoothness property is unfortunately lost for a nonoverlapping splitting.

We summarize, that domain decomposition algorithms with a theoretical guarantee to convergence to the minimizer of the global problem are till now given for (i) the discrete predual problem with a nonoverlapping decomposition [34], (ii) the continuous predual problem with an overlapping decomposition [13], and (iii) the discrete primal problem with a nonoverlapping decomposition [45].

In this paper, we continue growing this list by presenting convergent sequential and parallel overlapping domain decomposition methods for the primal problem (1.1). In particular we prove their convergence to the solution of (1.1) in an infinite dimensional setting. Thereby we follow basically the idea of [45]. That is, we consider the overlapping domain decomposition method in [13], which is stated in a continuous setting, and compute from its subdomain problems their respective dual problems. This yields an overlapping domain decomposition method for the problem in (1.1). Again, due to the connection of the subdomain problems to the predual subdomain problems of the method in [13], we are able to prove that our newly proposed methods converge to a minimizer of the underlying global problem. Although the proofs presented in this paper are motivated by the ones in [45], our analysis differs significantly from the one in [45]. This is due to the following reasons: Firstly, we consider overlapping domain decomposition methods, while in [45] nonoverlapping methods are considered. In particular, our subdomain problems are very different to the ones in [45], asking for different subdomain solvers as well as a different convergence analysis. Secondly, while the analysis in [45] is carried out in a discrete setting, our analysis as well as our methods are presented in a continuous setting, which generates additional difficulties in proving convergence. For example, while in a discrete setting bounded sequences have strongly converging subsequences, a bounded sequence in L^2 has (only) weakly converging subsequences. Moreover, the gradient of a bounded function is in general not bounded in L^2, while in a discrete setting the gradient of any bounded function is bounded again.

We found that in a discrete setting our proposed overlapping strategy allows for overlapping regions of size zero, yielding a nonoverlapping decomposition. In particular, in this limit case our proposed splitting strategies become the methods in [45], see Remark 3.1 below. Hence, in this sense our algorithms generalize the ones in [45].

The rest of the paper is organized as follows: In Section 2 we derive and present the proposed alternating and parallel domain decomposition methods. Their convergence to a minimizer of the global problem (1.1) is shown analytically. In Section 3 we describe two different approaches

on how to solve the subspace minimization problems. In particular, one approach first restricts the subspace problems to the subdomain and then discretizes the problem accordingly, while the other approach first discretizes the subproblems and than restricts it to the respective subdomain. Numerical experiments for the alternating and parallel domain decomposition methods are shown in Section 4, demonstrating the efficiency of the methods.

2. Overlapping Domain Decomposition Algorithms. In this section we present and analyze the proposed overlapping domain decomposition algorithms. Their derivation is based on the splitting strategy in [13]. However we start by fixing some notations.

2.1. Basic terminology. For a Banach space V we denote be V' its topological dual and $\langle \cdot, \cdot \rangle_{V' \times V}$ describes the bilinear canonical pairing over $V' \times V$. The norm of a Banach space V is written as $\| \cdot \|_V$. For any $\mathbf{p} = (p^1, p^2) \in \mathbb{L}^s(\Omega)$, $1 \leq s \leq \infty$, we define the norm $\|\mathbf{p}\|_{\mathbb{L}^s(\Omega)} = (\|p^1\|^2_{L^s(\Omega)} + \|p^2\|^2_{L^s(\Omega)})^{1/2}$. By $(\cdot, \cdot)$ we denote the standard inner product in $L^2(\Omega)$.

For ease of notation, in the sequel for any sequence $(v^n)_{n \in \mathbb{N}}$ we write $(v^n)_n$ instead. A convex functional $\mathcal{F} : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called *proper*, if $\{v \in V : \mathcal{F}(v) \neq +\infty\} \neq \emptyset$ and $\mathcal{F}(v) > -\infty$ for all $v \in V$. A functional $\mathcal{F} : V \to \overline{\mathbb{R}}$ is called *(weakly) lower semicontinuous (l.s.c.)*, if for every (weakly) convergent sequence $(v^n)_n \subset V$ with limit $v \in V$ we have

$$\liminf_{n \to \infty} \mathcal{F}(v^n) \geq \mathcal{F}(v).$$

Note, that in infinite dimensional spaces, weak l.s.c. is a stronger requirement than (strong) l.s.c. However, a convex and (strongly) l.s.c. function is weakly l.s.c. thanks to Mazur's lemma. Further, weak l.s.c. and (strong) l.s.c. coincide in finite dimensional spaces [5]. In the context of convergence we often use the symbols "$\to$" and "$\rightharpoonup$" to indicate strong and weak convergence respectively.

For a convex functional $\mathcal{F} : V \to \overline{\mathbb{R}}$ we define the *subdifferential* of $\mathcal{F}$ at $v \in V$ as the set valued function

$$\partial \mathcal{F}(v) := \begin{cases} \emptyset & \text{if } \mathcal{F}(v) = \infty, \\ \{v^* \in V' : \langle v^*, u - v \rangle_{V' \times V} + \mathcal{F}(v) \leq \mathcal{F}(u) \quad \forall u \in V\} & \text{otherwise.} \end{cases}$$

It is clear from this definition, that $0 \in \partial \mathcal{F}(v)$ if and only if v is a minimizer of $\mathcal{F}$.

The *conjugate function (or Legendre transform)* of a convex function $\mathcal{F} : V \to \overline{\mathbb{R}}$ is defined as $\mathcal{F}^* : V' \to \overline{\mathbb{R}}$ with

$$\mathcal{F}^*(v^*) = \sup_{v \in V} \{\langle v^*, v \rangle_{V' \times V} - \mathcal{F}(v)\}.$$

From this definition we see that $\mathcal{F}^*$ is the pointwise supremum of continuous affine functions and thus, according to [22, Proposition 3.1, p 14], convex, lower semicontinuous, and proper.

A functional $\mathcal{F} : V \to \overline{\mathbb{R}}$ is said to be *coercive* (in V), if for every sequence $(v^n)_n \subset V$ with $\|v^n\|_V \to \infty$, we have $\mathcal{F}(v^n) \to \infty$. Let V, W be two Banach spaces, then for any operator $\Lambda : V \to W$ we define by $\Lambda^* : W' \to V'$ its *adjoint*.

In the sequel we will often use $C(\Omega)$, the space of continuous functions in Ω, $C_0(\Omega, \mathbb{R}^2)$, the space of $\mathbb{R}^2$-valued continuous functions with compact support in Ω, and $\mathcal{D}(\Omega)$, the space of infinitely differentiable functions with compact support in Ω. By $H^1(\Omega)$ we denote the Sobolev space $W^{1,2}(\Omega)$, i.e., the space of functions in $L^2(\Omega)$, whose first weak derivatives are again in $L^2(\Omega)$.

2.2. Preliminaries. For ease of presentation, and in order to avoid unnecessary technicalities, we limit the derivation of our proposed domain decomposition algorithms to a splitting into 2 subdomains, by noting, that the generalization to multiple domains comes quite natural, see Section 2.5. Hence, we are considering an overlapping decomposition of the image domain Ω into 2 subdomains Ω_1 and Ω_2, such that $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. In order to compute a minimizer of (1.1) with respect to such a splitting, in [13] based on the predual formulation (1.2) the algorithm shown in Algorithm 2.1 is proposed.

Algorithm 2.1 Alternating predual version from [13]

    Initialize: $\mathbf{p}^0 \in H_0(\text{div}, \Omega)$ and $\hat{\alpha} \in (0, 1]$

    for $n = 0, 1, 2, \ldots$ **do**

        $\hat{\mathbf{q}}_1^n \in \arg\min\{\frac{1}{2}\|\text{div}(\mathbf{v} + \theta_2\mathbf{p}^n) + g\|_{L^2(\Omega)}^2 : \mathbf{v} \in H_0(\text{div}, \Omega), |\mathbf{v}(x)|_{\ell^2} \leq \alpha\theta_1(x)$ f.a.a. $x \in \Omega\}$

        $\mathbf{q}_1^n = (1 - \hat{\alpha})\theta_1\mathbf{p}^n + \hat{\alpha}\hat{\mathbf{q}}_1^n$

        $\hat{\mathbf{q}}_2^n \in \arg\min\{\frac{1}{2}\|\text{div}(\mathbf{v} + \mathbf{q}_1^n) + g\|_{L^2(\Omega)}^2 : \mathbf{v} \in H_0(\text{div}, \Omega), |\mathbf{v}(x)|_{\ell^2} \leq \alpha\theta_2(x)$ f.a.a. $x \in \Omega\}$

        $\mathbf{q}_2^n = (1 - \hat{\alpha})\theta_2\mathbf{p}^n + \hat{\alpha}\hat{\mathbf{q}}_2^n$

        $\mathbf{p}^{n+1} = (1 - \hat{\alpha})\mathbf{p}^n + \hat{\alpha}(\hat{\mathbf{q}}_1^n + \hat{\mathbf{q}}_2^n)$

    end for

    In Algorithm 2.1 $(\theta_i)_{i=1,2}$ denotes a partition of unity function with the properties

(a) $\theta_1 + \theta_2 \equiv 1$ and $\theta_i \geq 0$ a.e. on Ω,

(b) $\text{supp}\,\theta_i \subset \overline{\Omega}_i$,

(c) $\theta_i \in H^1(\Omega)$, $\|\nabla\theta_i\|_{\mathbb{L}^\infty(\Omega)} \leq c_0$,

for $i = 1, 2$, where $c_0 > 0$ is a constant depending on the overlapping-size; see [13]. Based on the relation (1.3) in [13] it is shown, that Algorithm 2.1 generates a sequence $(\mathbf{p}^n)_n$ such that $(u^n)_n$, where $u^n = \text{div}\,\mathbf{p}^n + g$, converges to a minimizer of (1.1).

    For deriving our algorithms, following the idea of [45], we calculate the dual problems of the subdomain problems of Algorithm 2.1. The subdomain problems in Algorithm 2.1 may be rewritten as

$$(2.1) \qquad \arg\min\left\{\frac{1}{2}\|\text{div}\,\mathbf{v} + f\|_{L^2(\Omega)}^2 : \mathbf{v} \in H_0(\text{div}, \Omega), |\mathbf{v}(x)|_{\ell^2} \leq \lambda(x) \text{ f.a.a. } x \in \Omega\right\}$$

where $f = \text{div}\,\theta_2\mathbf{p}^n + g$, $\lambda = \alpha\theta_1$ in Ω_1 and $f = \text{div}\,\mathbf{q}_1^n + g$, $\lambda = \alpha\theta_2$ in Ω_2 for any $n \geq 0$. Note that, since $f \in L^2(\Omega)$ in both situations and $\lambda: \overline{\Omega} \to \mathbb{R}_0^+$ is a bounded function, the existence of a minimizer of (2.1) is ensured [36, Proposition 3.2 (b)]. If $\lambda \in C(\overline{\Omega})$ and $\lambda(x) > 0$ for all $x \in \overline{\Omega}$, then a dual problem of (2.1) is given by

$$(2.2) \qquad \arg\min_{u \in L^2(\Omega)}\left\{\tilde{J}(u) := \frac{1}{2}\|u - f\|_{L^2(\Omega)}^2 + \int_\Omega \lambda|Du|\right\},$$

and has a unique solution, see [36, 37]. Here and in the sequel, the expression $\int_\Omega \lambda|Du|$ describes the integral of λ on Ω with respect to the measure $|Du|$, where Du is the distributional gradient of u, see [36] for more details. In this situation, thanks to [31, 36], we have the following relation between (2.1) and (2.2):

    LEMMA 2.1. *Let $f \in L^2(\Omega)$ be given and $\lambda: \overline{\Omega} \to \mathbb{R}^+$ with $\lambda \in C(\overline{\Omega})$ (i.e., λ is a positive continuous bounded function). Then u^* is a minimizer of (2.2) if and only if there exists a $\mathbf{p}^* \in H_0(\text{div}, \Omega)$ such that*

    (i) $\text{div}\,\mathbf{p}^* + f = u^*$ and

    (ii) $\mathbf{p}^* \in \arg\min_{\mathbf{p} \in K} \frac{1}{2}\|\text{div}\,\mathbf{p} + f\|_{L^2(\Omega)}^2$ where $K = K(\Omega) := \{\mathbf{p} \in H_0(\text{div}, \Omega): |\mathbf{p}(x)|_{\ell^2} \leq \lambda(x) \text{ f.a.a. } x \in \Omega\}$.

Further, $\mathbf{p}^* \in \arg\min_{\mathbf{p} \in K} \frac{1}{2}\|\text{div}\,\mathbf{p} + f\|_{L^2(\Omega)}^2$ if and only if

$$\langle(-\text{div})^*(\text{div}\,\mathbf{p}^* + f), \mathbf{p} - \mathbf{p}^*\rangle_{H_0(\text{div},\Omega)' \times H_0(\text{div},\Omega)} \leq 0 \quad \forall\mathbf{p} \in K$$

and $\mathbf{p}^* \in K$.

    In our case λ is not necessarily positive, i.e., $\lambda: \overline{\Omega} \to \mathbb{R}_0^+$, since it is in (2.2) the product of the regularization parameter α and a partition of unity function, i.e., $\lambda = \alpha\theta_i$ for $i \in \{1, 2\}$. This more general case is not covered in [36]. In particular, while in [36] the existence of minimizers of (2.1) with nonnegative, continuous and bounded λ is shown, no attention is given to problem (2.2) with such functions λ. Therefore, we show next the existence of a minimizer for (2.2) with $\lambda: \overline{\Omega} \to \mathbb{R}_0^+$ being continuous, i.e., λ is nonnegative, continuous and bounded. In the sequel we assume, that

$\mathrm{supp}(\lambda) \subseteq \overline{\Omega}$ and $\Omega_0 := \Omega \setminus \mathrm{supp}(\lambda)$ is then open. That is, if $\mathrm{supp}(\lambda) = \overline{\Omega}$, then $\Omega_0 = \emptyset$, otherwise, i.e., $\mathrm{supp}(\lambda) \subsetneq \overline{\Omega}$, then $\Omega_0 \neq \emptyset$. Since $\Omega_0 = \mathrm{int}(\Omega_0)$, it follows that if $\mathbf{q} \in C_0^1(\Omega, \mathbb{R}^2)$ and constant in Ω_0, i.e., $\mathbf{q}(x) = \mathbf{c} \in \mathbb{R}^2$ f.a.a. $x \in \Omega_0$, then $\mathrm{div}\, \mathbf{q} = 0$ a.e. on Ω_0.

LEMMA 2.2. *Let* $u \in L^2(\Omega)$, $\lambda : \overline{\Omega} \to \mathbb{R}_0^+$, $\lambda \in H^1(\Omega) \cap C(\overline{\Omega})$, $\|\nabla \lambda\|_{\mathbb{L}^\infty(\Omega)} < \infty$, *and* $\mathrm{supp}(\lambda) \subseteq \overline{\Omega}$, *then*

$$\sup_{\mathbf{p} \in K} (u, -\mathrm{div}\, \mathbf{p}) = \int_\Omega \lambda |Du| = \int_{\mathrm{supp}(\lambda)} \lambda |Du|.$$

Proof. We get

$$\int_{\mathrm{supp}(\lambda)} \lambda |Du| = \int_{\Omega \setminus \Omega_0} \lambda |Du| = \sup_{\mathbf{p} \in \mathcal{K}(1, C_0(\Omega \setminus \Omega_0, \mathbb{R}^2))} \langle \lambda Du, \mathbf{p} \rangle_{C_0(\Omega \setminus \Omega_0, \mathbb{R}^2)' \times C_0(\Omega \setminus \Omega_0, \mathbb{R}^2)}$$

$$= \sup_{\mathbf{p} \in \mathcal{K}(1, C_0(\Omega, \mathbb{R}^2))} \langle \lambda Du, \mathbf{p} \rangle_{C_0(\Omega, \mathbb{R}^2)' \times C_0(\Omega, \mathbb{R}^2)} = \int_\Omega \lambda |Du|,$$

since $\lambda \in C(\overline{\Omega})$ and $\lambda(x) = 0$ f.a.a. $x \in \Omega_0$, where $\mathcal{K}(\lambda, C_0(\Omega, \mathbb{R}^2)) := \{\mathbf{p} \in C_0(\Omega, \mathbb{R}^2) : |\mathbf{p}(x)|_{\ell^2} \leq \lambda(x) \text{ f.a.a. } x \in \Omega\}$. Since $\mathcal{K}(1, \mathcal{D}(\Omega)^2)$ is dense in $\mathcal{K}(1, C_0(\Omega, \mathbb{R}^2))$ in the sense of $C_0(\Omega, \mathbb{R}^2)$, and $\mathcal{K}(1, \mathcal{D}(\Omega)^2)$ is dense in $\mathcal{K}(1, H_0(\mathrm{div}, \Omega))$ in the sense of $H_0(\mathrm{div}, \Omega)$ [35], we observe

$$
\begin{aligned}
(2.3) \quad \int_\Omega \lambda |Du| &= \sup_{\mathbf{p} \in \mathcal{K}(1, C_0(\Omega, \mathbb{R}^2))} \langle \lambda Du, \mathbf{p} \rangle_{C_0(\Omega, \mathbb{R}^2)' \times C_0(\Omega, \mathbb{R}^2)} \\
&= \sup_{\mathbf{p} \in \mathcal{K}(1, \mathcal{D}(\Omega)^2)} \langle \lambda Du, \mathbf{p} \rangle_{C_0(\Omega, \mathbb{R}^2)' \times C_0(\Omega, \mathbb{R}^2)} \\
&= \sup_{\mathbf{p} \in \mathcal{K}(1, \mathcal{D}(\Omega)^2)} \langle Du, \lambda \mathbf{p} \rangle_{C_0(\Omega, \mathbb{R}^2)' \times C_0(\Omega, \mathbb{R}^2)} = \sup_{\mathbf{p} \in \mathcal{K}(1, \mathcal{D}(\Omega)^2)} (u, -\mathrm{div}\, \lambda \mathbf{p}) \\
&= \sup_{\mathbf{p} \in \mathcal{K}(1, H_0(\mathrm{div}, \Omega))} (u, -\mathrm{div}\, \lambda \mathbf{p}) \leq \sup_{\mathbf{p} \in K} (u, -\mathrm{div}\, \mathbf{p}),
\end{aligned}
$$

where we note that $\lambda \mathbf{p} \in H_0(\mathrm{div}, \Omega)$ for any $\mathbf{p} \in H_0(\mathrm{div}, \Omega)$, due to the assumptions on λ. On the other hand, for any $\epsilon > 0$, using the same argument as above together with $(\lambda + \epsilon)\mathbf{p} \in H_0(\mathrm{div}, \Omega)$ for any $\mathbf{p} \in H_0(\mathrm{div}, \Omega)$, we obtain $\int_\Omega (\lambda + \epsilon)|Du| = \sup_{\mathbf{p} \in K_\epsilon} (u, -\mathrm{div}\, \lambda \mathbf{p}) \geq \sup_{\mathbf{p} \in K} (u, -\mathrm{div}\, \mathbf{p})$, where $K_\epsilon := \{p \in H_0(\mathrm{div}, \Omega) : |\mathbf{p}| \leq \lambda + \epsilon\}$ and the equality follows by [36]. Letting ϵ tend to 0, shows $\int_\Omega \lambda |Du| \geq \sup_{\mathbf{p} \in K} (u, -\mathrm{div}\, \mathbf{p})$. This together with (2.3) proves the assertion. □

Now we prove the existence and uniqueness of the minimizer of (2.2).

PROPOSITION 2.3. *If* λ *is defined as in Lemma 2.2, then* (2.2) *admits a unique solution.*

Proof. The functional $\tilde{J}$ is obviously bounded from below by 0 and there exists a $v \in L^2(\Omega)$ such that $\tilde{J}(v) \in \mathbb{R}$. Further, $\tilde{J}$ is coercive in $L^2(\Omega)$, since $\|u - f\|_{L^2(\Omega)}^2 \leq \tilde{J}(u)$ for all $u \in L^2(\Omega)$. The weak lower semicontinuity of $\tilde{J}$ follows by the continuity of the map $u \to \int_\Omega -\mathrm{div}\, \mathbf{p}\, u\, \mathrm{d}x$, where $\mathbf{p} \in K$. Then the direct method in the calculus of variations yields the existence of a minimizer.

The uniqueness of the minimizer follows by the strict convexity of $\tilde{J}$. □

Note, that a minimizer of $\tilde{J}$ not necessarily is an element of $BV(\Omega)$. This is easily seen, since a minimizer is allowed to have infinite variation in Ω_0. In order to show the duality relation between (2.1) and (2.2) for $\lambda \geq 0$, we recall the Fenchel duality theorem; see, e.g., [22] for more details.

THEOREM 2.4 (Fenchel duality theorem). *Let* V *and* W *be two Banach spaces with topological duals* V' *and* W', *respectively, and* $\Lambda : V \to W$ *a bounded linear operator with adjoint* $\Lambda^* \in \mathcal{L}(W', V')$. *Further let* $\mathcal{F} : V \to \overline{\mathbb{R}}$, $\mathcal{G} : W \to \overline{\mathbb{R}}$ *be convex, lower semicontinuous, and proper functionals. Assume there exists* $q_0 \in V$ *such that* $\mathcal{F}(q_0) < \infty$, $\mathcal{G}(\Lambda q_0) < \infty$ *and* $\mathcal{G}$ *is continuous at* Λq_0. *Then we have*

$$(2.4) \qquad \inf_{q \in V} \mathcal{F}(q) + \mathcal{G}(\Lambda q) = \sup_{v \in W'} -\mathcal{F}^*(\Lambda^* v) - \mathcal{G}^*(-v)$$

and the problem on the right hand side of (2.4) admits a solution $\bar{v}$. Moreover, $\bar{v}$ and $\bar{q}$ are solutions of the two optimization problems in (2.4), respectively, if and only if

$$
\begin{aligned}
(2.5) \quad & \Lambda^*\bar{v} \in \partial\mathcal{F}(\bar{q}), \\
& -\bar{v} \in \partial\mathcal{G}(\Lambda\bar{q}).
\end{aligned}
$$

Now we are able to state our duality result.

THEOREM 2.5. *Let λ be defined as in Lemma 2.2. Then a Fenchel dual of (2.1) is given by (2.2). Moreover, let u^* be a minimizer of (2.2) and $\mathbf{p}^*$ a minimizer of (2.1) then*

$$
(2.6) \qquad u^* = \operatorname{div}\mathbf{p}^* + f \qquad and \qquad \langle(-\operatorname{div})^* u^*, \mathbf{p} - \mathbf{p}^*\rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p} \in K.
$$

Proof. We apply the Fenchel duality result (see Theorem 2.4) with $V = H_0(\operatorname{div},\Omega)$, $W = L^2(\Omega)$, $\Lambda = -\operatorname{div}$, $\mathcal{F} : V \to \mathbb{R}$ given by $\mathcal{F}(\mathbf{p}) = \mathbf{I}_K(\mathbf{p})$, and $\mathcal{G} : L^2(\Omega) \to \mathbb{R}$ given by $\mathcal{G}(v) = \frac{1}{2}\|v - f\|_{L^2(\Omega)}^2$, where

$$
\mathbf{I}_K(\mathbf{p}) = \begin{cases} 0 & \text{if } \mathbf{p} \in K \\ \infty & \text{otherwise} \end{cases}.
$$

The convex conjugate $\mathcal{G}^* : L^2(\Omega) \to \mathbb{R}$ is then $\mathcal{G}^*(v) = \frac{1}{2}\|v + f\|_{L^2(\Omega)}^2 - \frac{1}{2}\|f\|_{L^2(\Omega)}$, cf. [31], and the convex conjugate $\mathcal{F}^* : V' \to \mathbb{R}$ is given by $\mathcal{F}^*(\mathbf{q}) = \sup_{\mathbf{p}\in V}\langle\mathbf{q},\mathbf{p}\rangle_{V'\times V} - \mathcal{F}(\mathbf{p})$. Thus

$$
\mathcal{F}^*(\Lambda^*v) = \sup_{\mathbf{p}\in K}(v, -\operatorname{div}\mathbf{p}) = \int_\Omega \lambda|Dv|,
$$

where we used Lemma 2.2. From (2.5) we find (2.6). $\qquad\square$

In order that (2.2) is well defined and that convergence to the true minimizer of the proposed domain decomposition algorithms is guaranteed, a partition of unity function needs to have the following properties

$$
(2.7) \qquad \text{(a') } \theta_1 + \theta_2 \equiv 1 \text{ and } \theta_i \geq 0 \text{ a.e. on } \overline{\Omega} \text{ for } i = 1, 2,
$$

$$
(2.8) \qquad \text{(b') } \operatorname{supp}(\theta_i) \subset \overline{\Omega}_i \text{ for } i = 1, 2,
$$

$$
(2.9) \qquad \text{(c') } \theta_i \in H^1(\Omega) \cap C(\overline{\Omega}) \text{ and } \|\nabla\theta_i\|_{\mathbb{L}^\infty(\Omega)} < \infty \text{ for } i = 1, 2.
$$

In the sequel we will only use a partition of unity function with the properties (a'), (b'), and (c'), and denote it by $(\theta_i)_i$.

Remark 2.6. The theory presented in this paper can be generalized to other types of total variation, i.e.,

$$
\int_\Omega |Du|_r = \sup\left\{ \int_\Omega u\operatorname{div}\mathbf{p}\,\mathrm{d}x \ : \ \mathbf{p} \in C_0^1(\Omega, \mathbb{R}^2), |\mathbf{p}|_{\ell^r} \leq 1 \text{ a.e. in } \Omega \right\}
$$

with $1 \leq r < +\infty$, leading to a similar analysis and similar results; cf. Remark 2.16 below. For instance, the predual problem of (2.2) with $\int_\Omega |Du|_r$ is given by

$$
\operatorname{arg\,min}\left\{ \frac{1}{2}\|\operatorname{div}\mathbf{v} + f\|_{L^2(\Omega)}^2 : \mathbf{v} \in H_0(\operatorname{div},\Omega), |\mathbf{v}(x)|_{\ell^{r^*}} \leq \lambda(x) \text{ f.a.a. } x \in \Omega \right\},
$$

where $\frac{1}{r} + \frac{1}{r^*} = 1$.

2.3. Alternating algorithm. Based on the above considerations we are now able to present the proposed alternating domain decomposition algorithm, stated in Algorithm 2.2.

LEMMA 2.7. *The sequence $(u^n)_n$ as well as $(u_i^n)_n$, $i = 1, 2$, generated by Algorithm 2.2 are bounded in $L^2(\Omega)$. Moreover, we have*

$$
(2.10) \qquad \|u_2^n\|_{L^2(\Omega)} \geq \|u_1^{n+1}\|_{L^2(\Omega)} \geq \|u_2^{n+1}\|_{L^2(\Omega)} \geq \|u_1^{n+2}\|_{L^2(\Omega)} \qquad for \text{ all } n \geq 1.
$$

Algorithm 2.2 Alternating Version

Initialize: $u_2^0(=0) \in L^2(\Omega), f_2^0 = 0 \in L^2(\Omega)$
for $n = 0, 1, 2, \ldots$ **do**
$\quad f_1^{n+1} = u_2^n - f_2^n + g$
$\quad u_1^{n+1} = \arg\min_{u_1 \in L^2(\Omega)} \frac{1}{2}\|u_1 - f_1^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_1 |Du_1|$
$\quad f_2^{n+1} = u_1^{n+1} - f_1^{n+1} + g$
$\quad u_2^{n+1} = \arg\min_{u_2 \in L^2(\Omega)} \frac{1}{2}\|u_2 - f_2^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_2 |Du_2|$
$\quad u^{n+1} = g - f_1^{n+1} - f_2^{n+1} + u_1^{n+1} + u_2^{n+1}(= u_2^{n+1})$
end for

Proof. By the relation to the predual problem for $n \geq 1$ we have for $i = 1, 2$, that there exist $\mathbf{p}_i^n \in K_i := \{\mathbf{p} \in H_0(\mathrm{div}, \Omega) : |\mathbf{p}|_{\ell^2} \leq \alpha\theta_i \text{ a.e. on } \Omega\}$ such that

$$(2.11) \qquad u_i^n = \mathrm{div}\,\mathbf{p}_i^n + f_i^n \quad \text{and} \quad \langle(-\mathrm{div})^* u_i^n, \mathbf{p}_i - \mathbf{p}_i^n\rangle_{H_0(\mathrm{div},\Omega)' \times H_0(\mathrm{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_i \in K_i.$$

This means that for all $n \geq 1$ and for $i = 1, 2$ such $\mathbf{p}_i^n \in K_i$ solves

$$\arg\min_{\mathbf{p} \in K_i} \frac{1}{2}\|\mathrm{div}\,\mathbf{p} + f_i^n\|_{L^2(\Omega)}^2.$$

Since $f_1^{n+1} = u_2^n - f_2^n + g = \mathrm{div}\,\mathbf{p}_2^n + g$ and $f_2^{n+1} = u_1^{n+1} - f_1^{n+1} + g = \mathrm{div}\,\mathbf{p}_1^{n+1} + g$ we have

$$\mathbf{p}_1^{n+1} \in \arg\min_{\mathbf{p}_1 \in K_1} \frac{1}{2}\|\mathrm{div}\,\mathbf{p}_1 + \mathrm{div}\,\mathbf{p}_2^n + g\|_{L^2(\Omega)}^2 \quad \text{and} \quad \mathbf{p}_2^{n+1} \in \arg\min_{\mathbf{p}_2 \in K_2} \frac{1}{2}\|\mathrm{div}\,\mathbf{p}_2 + \mathrm{div}\,\mathbf{p}_1^{n+1} + g\|_{L^2(\Omega)}^2$$

and hence
(2.12)
$$\frac{1}{2}\|\mathrm{div}\,\mathbf{p}_1^n + \mathrm{div}\,\mathbf{p}_2^n + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\mathrm{div}\,\mathbf{p}_1^{n+1} + \mathrm{div}\,\mathbf{p}_2^n + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\mathrm{div}\,\mathbf{p}_1^{n+1} + \mathrm{div}\,\mathbf{p}_2^{n+1} + g\|_{L^2(\Omega)}^2$$

for all $n \geq 1$. Set $\mathbf{p}^n := \mathbf{p}_1^n + \mathbf{p}_2^n$ and $\tilde{\mathbf{p}}^{n+1} := \mathbf{p}_1^{n+1} + \mathbf{p}_2^n$ for all $n \geq 1$, then (2.12) yields

$$\frac{1}{2}\|\mathrm{div}\,\mathbf{p}^1 + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\mathrm{div}\,\tilde{\mathbf{p}}^{n+1} + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\mathrm{div}\,\mathbf{p}^{n+1} + g\|_{L^2(\Omega)}^2 \qquad \forall n \geq 1.$$

Note that $\mathbf{p}^1 \in H_0(\mathrm{div}, \Omega)$ and hence $\|\mathrm{div}\,\mathbf{p}^1 + g\|_{L^2(\Omega)}^2 < \infty$. Then, by monotonicity we obtain

$$\|\mathrm{div}\,\tilde{\mathbf{p}}^{n+1} + g\|_{L^2(\Omega)} < \infty \qquad \text{and} \qquad \|\mathrm{div}\,\mathbf{p}^n + g\|_{L^2(\Omega)} < \infty$$

for all $n \geq 1$. Moreover,

$$\|\mathbf{p}_i^n\|_{\mathbb{L}^2(\Omega)}^2 = \int_\Omega |p_i^{1,n}(x)|^2 \mathrm{d}x + \int_\Omega |p_i^{2,n}(x)|^2 \mathrm{d}x \leq \alpha^2 \int_\Omega |\theta_i(x)|^2 \mathrm{d}x < \infty,$$

where $\mathbf{p}_i^n = (p_i^{1,n}, p_i^{2,n})$, and hence

$$\|\mathbf{p}^n\|_{\mathbb{L}^2(\Omega)} \leq \|\mathbf{p}_1^n\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{p}_2^n\|_{\mathbb{L}^2(\Omega)} < \infty \quad \text{and} \quad \|\tilde{\mathbf{p}}^{n+1}\|_{\mathbb{L}^2(\Omega)} < \infty$$

for all $n \geq 1$, which shows the boundedness of $(\mathbf{p}^n)_n$ and $(\tilde{\mathbf{p}}^{n+1})_n$ in $H_0(\mathrm{div}, \Omega)$. Set $f^{n+1} := f_1^{n+1} + f_2^{n+1} = \mathrm{div}\,\tilde{\mathbf{p}}^{n+1} + 2g$ and $\tilde{u}^n := u_1^n + u_2^n = \mathrm{div}\,\mathbf{p}^n + f^n$ for all $n \geq 1$, then we have that $(f^n)_n$ and $(\tilde{u}^n)_n$ are bounded in $L^2(\Omega)$. Consequently, the sequences $(u^n)_n$ and $(u_2^n)_n$, see Algorithm 2.2, are bounded in $L^2(\Omega)$. Moreover, we get the boundedness of $(u_1^n)_n$, which is easily seen from the definition of $\tilde{u}^n$.

Due to the relation (2.11) we obtain from the monotonicity (2.12) that

$$\|u_1^n - f_1^n + u_2^n - f_2^n + g\|_{L^2(\Omega)}^2 \geq \|u_1^{n+1} - f_1^{n+1} + u_2^n - f_2^n + g\|_{L^2(\Omega)}^2$$
$$\geq \|u_1^{n+1} - f_1^{n+1} + u_2^{n+1} - f_2^{n+1} + g\|_{L^2(\Omega)}^2$$

for all $n \geq 1$. Since we have $u_2^n - f_2^n - f_1^{n+1} + g = 0$ and $u_1^n - f_2^n - f_1^n + g = 0$ for all $n \geq 1$, see Algorithm 2.2, the latter inequalities yield (2.10). $\qquad\square$

Since $u^n = u_2^n$ for all $n \geq 1$, the previous lemma shows, that $(u^n)_n$ possesses weakly converging subsequences. Moreover, from the monotonicity in Lemma 2.7 we even get that the whole sequence $(\|u^n\|_{L^2(\Omega)})_n$ converges monotonically to some limit. Note, that $\|u^n\|_{L^2(\Omega)} = \|u^{n+1}\|_{L^2(\Omega)}$ does not imply $u^n = u^{n+1}$. Now, we are wondering if the weak limit of such a subsequence indeed solves the global minimization problem (1.1). In order to argue that, we need the boundedness of $(f_i^n)_n$ in $L^2(\Omega)$. By Algorithm 2.2 we have

$$f_1^n = \sum_{k=1}^{n-1}(u_2^k - u_1^k) + u_2^0 - f_2^0 + g, \qquad f_2^n = u_1^n + \sum_{k=1}^{n-1}(u_1^k - u_2^k) - u_2^0 + f_2^0 = \sum_{k=1}^{n-1}(u_1^{k+1} - u_2^k) + u_1^1$$

for all $n \geq 1$. Hence, if there exists a constant $C > 0$, which is independent of n such that $\sum_{k=1}^{n-1}\|u_2^k - u_1^k\|_{L^2(\Omega)} \leq C < \infty$ for any $n \geq 1$ (or in other words, if $\|u_2^k - u_1^k\|_{L^2(\Omega)}$ converges to zero fast enough for $k \to \infty$, for instance, $\|u_2^k - u_1^k\|_{L^2(\Omega)} \leq \frac{1}{k^\gamma}$ with $\gamma > 1$), then $(f_i^n)_n$ is bounded in $L^2(\Omega)$. Unfortunately, in the sequel we are only able to show that if $(f_i^n)_n$ is bounded, then $\lim_{n\to\infty}\|u_2^n - u_1^n\|_{L^2(\Omega)} = 0$.

THEOREM 2.8. *If there is an $i \in \{1,2\}$ such that $(f_i^n)_n$ is bounded in $L^2(\Omega)$, then Algorithm 2.2 generates a sequence $(u^n)_n$ which converges strongly in $L^2(\Omega)$ to the unique minimizer of (1.1).*

Proof. We use the notation of the previous proof.

Since $(f_i^n)_n$ is bounded in $L^2(\Omega)$ for one $i \in \{1,2\}$, we get by the proof of Lemma 2.7 and Algorithm 2.2 that also $(f_{i^c}^n)$, $i^c \in \{1,2\} \setminus \{i\}$ is bounded. Then (2.11) implies that $(\operatorname{div} \mathbf{p}_1^n)_n$ and $(\operatorname{div} \mathbf{p}_2^n)_n$ are bounded in $L^2(\Omega)$ and hence $(\mathbf{p}_i^n)_n$ is bounded in $H_0(\operatorname{div}, \Omega)$.

By the boundedness of $(\mathbf{p}_2^n)_n$ there is a subsequence $(\mathbf{p}_2^{n_k})_k$ which converges weakly to a limit denoted by $\mathbf{p}_2^\infty$. Since $(\mathbf{p}_2^{n_k-1})_k$ is bounded as well, there exists a subsequence $(\mathbf{p}_2^{n_{k_j}-1})_j$ with weak limit $\tilde{\mathbf{p}}_2^\infty$. Further $(\mathbf{p}_1^{n_{k_j}})_j$ is bounded and has a subsequence $(\mathbf{p}_1^{n_{k_{j_l}}})$ which converges weakly to $\mathbf{p}_1^\infty$. Thus, from

$$\frac{1}{2}\|\operatorname{div} \mathbf{p}_1^{n-1} + \operatorname{div} \mathbf{p}_2^{n-1} + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\operatorname{div} \mathbf{p}_1^n + \operatorname{div} \mathbf{p}_2^{n-1} + g\|_{L^2(\Omega)}^2 \geq \frac{1}{2}\|\operatorname{div} \mathbf{p}_1^n + \operatorname{div} \mathbf{p}_2^n + g\|_{L^2(\Omega)}^2 \geq 0,$$

for all $n \geq 2$, we get that

$$\frac{1}{2}\|\operatorname{div} \mathbf{p}_1^\infty + \operatorname{div} \mathbf{p}_2^\infty + g\|_{L^2(\Omega)}^2 = \frac{1}{2}\|\operatorname{div} \mathbf{p}_1^\infty + \operatorname{div} \tilde{\mathbf{p}}_2^\infty + g\|_{L^2(\Omega)}^2.$$

By the optimality $\mathbf{p}_2^\infty \in \arg\min_{\mathbf{p}_2 \in K_2} \frac{1}{2}\|\operatorname{div} \mathbf{p}_1^\infty + \operatorname{div} \mathbf{p}_2 + g\|_{L^2(\Omega)}^2$ and the strict convexity of $\|\cdot\|_{L^2(\Omega)}^2$ we obtain

$$(2.13) \qquad\qquad \operatorname{div} \tilde{\mathbf{p}}^\infty = \operatorname{div} \mathbf{p}^\infty,$$

where $\mathbf{p}^\infty = \mathbf{p}_1^\infty + \mathbf{p}_2^\infty$ and $\tilde{\mathbf{p}}^\infty = \mathbf{p}_1^\infty + \tilde{\mathbf{p}}_2^\infty$.

We show now that $(u^n)_n$ has a weak accumulation point minimizing (1.1). A function $u^* \in L^2(\Omega)$ solves (1.1) if and only if there exists $\mathbf{p}^* \in H_0(\operatorname{div}, \Omega)$ with $|\mathbf{p}^*(x)|_{\ell^2} \leq \alpha$ f.a.a. $x \in \Omega$ such that
  (i) $\operatorname{div} \mathbf{p}^* + g = u^*$ and
  (ii) $\langle(-\operatorname{div})^*(\operatorname{div} \mathbf{p}^* + g), \mathbf{p} - \mathbf{p}^*\rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0$ for all $\mathbf{p} \in H_0(\operatorname{div}, \Omega)$ with $|\mathbf{p}(x)|_{\ell^2} \leq \alpha$ f.a.a. $x \in \Omega$.

By the equality in (2.11) we have that

$$u_1^n + u_2^n = \operatorname{div} \mathbf{p}_1^n + \operatorname{div} \mathbf{p}_2^n + f_1^n + f_2^n,$$

and hence

$$u^n = u_1^n + u_2^n - f_1^n - f_2^n + g = g + \operatorname{div} \mathbf{p}^n$$

for all $n \geq 1$. Since all occurring sequences are bounded we are able to take suitable subsequences with weak limits u^∞ and $\mathbf{p}^\infty$ and obtain

$$(2.14) \qquad\qquad u^\infty = g + \operatorname{div} \mathbf{p}^\infty.$$

By the inequality in (2.11) for $n \geq 2$ we have that

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_1^n + f_1^n), \mathbf{p}_1 - \mathbf{p}_1^n \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_2^n + f_2^n), \mathbf{p}_2 - \mathbf{p}_2^n \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2,$$

which is equivalent to

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_1^n + \operatorname{div} \mathbf{p}_2^{n-1} + g), \mathbf{p}_1 - \mathbf{p}_1^n \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_2^n + \operatorname{div} \mathbf{p}_1^n + g), \mathbf{p}_2 - \mathbf{p}_2^n \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2.$$

Since all quantities are bounded, we take suitable subsequences n_k, such that $\mathbf{p}_i^{n_k} \rightharpoonup \mathbf{p}_i^\infty$ for $i = \{1, 2\}$ and $\tilde{\mathbf{p}}^{n_k} \rightharpoonup \tilde{\mathbf{p}}^\infty$, and we get

$$\langle (-\operatorname{div})^*(\operatorname{div} \tilde{\mathbf{p}}^\infty + g), \mathbf{p}_1 - \mathbf{p}_1^\infty \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^\infty + g), \mathbf{p}_2 - \mathbf{p}_2^\infty \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2.$$

By (2.13) and summing up the latter two inequalities yields

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^\infty + g), \mathbf{p} - \mathbf{p}^\infty \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p} := \mathbf{p}_1 + \mathbf{p}_2 \in K_1 + K_2.$$

Note, that $|\mathbf{p}|_{\ell^2} = |\mathbf{p}_1 + \mathbf{p}_2|_{\ell^2} \leq |\mathbf{p}_1|_{\ell^2} + |\mathbf{p}_2|_{\ell^2} \leq \alpha$ a.e. on Ω. On the contrary for any $\mathbf{p} \in H_0(\operatorname{div},\Omega)$ with $|\mathbf{p}|_{\ell^2} \leq \alpha$ we set $\mathbf{p}_i = \theta_i \mathbf{p}$ for $i = \{1, 2\}$, which implies that $|\mathbf{p}_i|_{\ell^2} \leq \alpha \theta_i$ and $\mathbf{p}_i \in K_i$, thanks to (2.7)–(2.9). Hence we obtain

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^\infty + g), \mathbf{p} - \mathbf{p}^\infty \rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p} \in H_0(\operatorname{div},\Omega) \text{ with } |\mathbf{p}|_{\ell^2} \leq \alpha \text{ a.e. on } \Omega.$$

This together with (2.14) and the fact that $|\mathbf{p}^\infty|_{\ell^2} \leq \alpha$ a.e. on Ω shows that u^∞ is a minimizer of (1.1). Since any converging subsequence of $(u^n)_n$ converges weakly in $L^2(\Omega)$ to the unique minimizer of (1.1), there is only one accumulation point and hence $u^n \rightharpoonup u^\infty$.

From the monotonicity (2.10) we get that $\|u^n\|_{L^2(\Omega)} \to \|u^\infty\|_{L^2(\Omega)}$ monotonically. This together with the weak convergence $u^n \rightharpoonup u^\infty$ yields the assertion, since $\|u^n\|_{L^2(\Omega)}^2 = \|u^n - u^\infty\|_{L^2(\Omega)}^2 + 2(u^n - u^\infty, u^\infty) + \|u^\infty\|_{L^2(\Omega)}^2$. $\square$

Remark 2.9. The above proof relies on the boundedness of $(\mathbf{p}_i^n)_n$ in $H_0(\operatorname{div}, \Omega)$, which is attained if $(f_i^n)_n$ is bounded in $L^2(\Omega)$ for $i = \{1, 2\}$. However, in a finite dimensional setting, which is for example the situation when the considered problem is discretized, the boundedness of $(\mathbf{p}_i^n)_n$ implies the boundedness of $(\operatorname{div} \mathbf{p}_i^n)_n$ for $i = 1, 2$, which in turn yields the boundedness of $(f_i^n)_n$. Hence, in this situation the boundedness assumption on $(f_i^n)_n$ always holds and thus can be dropped in Theorem 2.8. Consequently, in a discrete setting by the above considerations we can get a vague idea of the convergence order. In particular, there exists a $\gamma > 1$ such that $\|u_1^n - u_2^n\|_X \leq \frac{1}{n^\gamma}$ for all $n \geq 1$, where $\|\cdot\|_X$ defines an appropriate discrete norm.

We emphasize, Theorem 2.8 shows, that the whole sequence $(u^n)_n$ generated by Algorithm 2.2 converges strongly to a minimizer of the global problem (1.1), thanks to the monotonicity property in Lemma 2.7. This monotonicity property also guarantees, that $(u_1^n)_n$ converges strongly to the minimizer of (1.1), which we show next.

COROLLARY 2.10. *Suppose the assumption of Theorem 2.8 holds. Then $(u_1^n)_n$ generated by Algorithm 2.2 converges strongly in $L^2(\Omega)$ to the minimizer of (1.1).*

Proof. Since $(u_1^n)_n$ is bounded in $L^2(\Omega)$ and $u_1^n = \operatorname{div} \tilde{\mathbf{p}}^n + g$ for all $n \geq 2$, there exists a subsequence such that $u_1^{n_k} \rightharpoonup u_1^\infty := \operatorname{div} \tilde{\mathbf{p}}^\infty + g = \operatorname{div} \mathbf{p}^\infty + g = u^\infty$ (in $L^2(\Omega)$), where we used (2.13). Since this is true for any convergent subsequence, we get $u_1^n \rightharpoonup u^\infty$ for $n \to \infty$. The strong convergence follows from the monotonicity (2.10) together with the weak convergence. $\square$

COROLLARY 2.11. *Suppose the assumption of Theorem 2.8 holds. Then $(f_i^n)_n$, $i = 1, 2$, generated by Algorithm 2.2 converges strongly in $L^2(\Omega)$.*

Proof. By Algorithm 2.2 we have for all $n \geq 0$,

$$f_1^{n+1} = u_2^n - f_2^n + g = u_2^n - u_1^n + f_1^n, \quad f_2^{n+1} = u_1^{n+1} - f_1^{n+1} + g = u_1^n - u_2^n + f_2^n$$

and hence

$$f_1^{n+1} - f_1^n = u_2^n - u_1^n, \qquad f_2^{n+1} - f_2^n = u_1^{n+1} - u_2^n.$$

This together with Corollary 2.10 and the boundedness of $(f_i^n)_n$, $i = 1, 2$, implies the assertion. □

2.4. Parallel algorithm. The parallel version of the domain decomposition algorithm in Algorithm 2.2 is presented in Algorithm 2.3. We state a similar convergence result as for the sequential algorithm.

Algorithm 2.3 Parallel Version

Initialize: $v_i^0 = 0$ for $i = 1, 2$
for $n = 0, 1, 2, \ldots$ **do**
  $f_1^{n+1} = v_2^n + g$
  $f_2^{n+1} = v_1^n + g$
  $u_1^{n+1} = \arg\min_{u_1 \in L^2(\Omega)} \frac{1}{2}\|u_1 - f_1^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_1 |Du_1|$
  $u_2^{n+1} = \arg\min_{u_2 \in L^2(\Omega)} \frac{1}{2}\|u_2 - f_2^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_2 |Du_2|$
  $v_1^{n+1} = \frac{v_1^n + u_1^{n+1} - f_1^{n+1}}{2}$
  $v_2^{n+1} = \frac{v_2^n + u_2^{n+1} - f_2^{n+1}}{2}$
  $u^{n+1} = g + v_1^{n+1} + v_2^{n+1} (= \frac{u_1^{n+1} + u_2^{n+1}}{2})$
end for

THEOREM 2.12. *Assume that $(f_i^n)_n$ is bounded in $L^2(\Omega)$ for $i = 1, 2$. Then Algorithm 2.3 generates a sequence $(u^n)_n$, which converges strongly in $L^2(\Omega)$ to the minimizer of (1.1).*

Proof. For the sake of clarity we present the proof in four steps.

Step 1: One shows by induction, as in [45, Proof of Lemma 3.4], that for $i = 1, 2$ there exist $\tilde{\mathbf{v}}_i^n \in K_i$ such that div $\tilde{\mathbf{v}}_i^n = v_i^n$ for all $n \geq 0$.

Step 2: Now let us show that $(\mathbf{p}_i^{n+1})_n$ is bounded in $H_0(\text{div}, \Omega)$. From the boundedness of $(f_i^n)_n$ we directly get that $(v_i^n)_n$ is bounded. We note that for all $n \geq 0$

$$\mathbf{p}_i^{n+1} \in \underset{p_i \in K_i}{\arg\min} \|\text{div}\, \mathbf{p}_i + f_i^{n+1}\|_{L^2(\Omega)}^2,$$

where f_i^{n+1} is given by Algorithm 2.3. Using this and the triangle inequality, we obtain

$$
\begin{aligned}
\|v_1^{n+1} + v_2^{n+1} + g\|_{L^2(\Omega)} &= \left\|\frac{v_1^n + \text{div}\, \mathbf{p}_1^{n+1}}{2} + \frac{v_2^n + \text{div}\, \mathbf{p}_2^{n+1}}{2} + g\right\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}\left(\|\text{div}\, \mathbf{p}_1^{n+1} + v_2^n + g\|_{L^2(\Omega)} + \|v_1^n + \text{div}\, \mathbf{p}_2^{n+1} + g\|_{L^2(\Omega)}\right) \\
&\leq \frac{1}{2}\left(\|\text{div}\, \tilde{\mathbf{v}}_1^n + v_2^n + g\|_{L^2(\Omega)} + \|v_1^n + \text{div}\, \tilde{\mathbf{v}}_2^n + g\|_{L^2(\Omega)}\right) \\
&= \|v_1^n + v_2^n + g\|_{L^2(\Omega)}
\end{aligned}
$$

(2.15)

for all $n \geq 0$. Similarly one gets

$$
\begin{aligned}
\|v_1^{n+1} + v_2^n + g\|_{L^2(\Omega)} &= \left\|\frac{v_1^n + \text{div}\, \mathbf{p}_1^{n+1}}{2} + v_2^n + g\right\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}\left(\|\text{div}\, \mathbf{p}_1^{n+1} + v_2^n + g\|_{L^2(\Omega)} + \|v_1^n + v_2^n + g\|_{L^2(\Omega)}\right) \\
&= \|v_1^n + v_2^n + g\|_{L^2(\Omega)}
\end{aligned}
$$

(2.16)

for all $n \geq 0$. By the monotonicity in (2.15) we have that $\|v_1^n + v_2^n + g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$. Thus from (2.16) we obtain the boundedness in $L^2(\Omega)$ for $(\operatorname{div} \mathbf{p}_1^{n+1})_n$, since

$$\|g\|_{L^2(\Omega)} \geq \|v_1^n + v_2^n + g\|_{L^2(\Omega)} \geq \|\operatorname{div} \mathbf{p}_1^{n+1} + v_2^n + g\|_{L^2(\Omega)} \geq \|\operatorname{div} \mathbf{p}_1^{n+1}\|_{L^2(\Omega)} - \|v_2^n\|_{L^2(\Omega)} - \|g\|_{L^2(\Omega)},$$

for all $n \geq 0$. An analog argument yields the boundedness of $(\operatorname{div} \mathbf{p}_2^{n+1})_n$ in $L^2(\Omega)$.

Since $\mathbf{p}_i^{n+1} \in K_i$ for all $n \geq 0$, we get $\|\mathbf{p}_i^{n+1}\|_{\mathbb{L}^2(\Omega)}^2 \leq \alpha^2 \|\theta_i\|_{L^2(\Omega)}^2 < \infty$ and hence $(\mathbf{p}_i^{n+1})_n$ is bounded in $H_0(\operatorname{div}, \Omega)$.

Step 3: We show that weak accumulation points of $(v_i^n)_n$ and $(\operatorname{div} \mathbf{p}_i^{n+1})_n$ coincide. Since both sequences are bounded, there exist suitable subsequences such that $(v_i^{n_k})_k$ and $(\operatorname{div} \mathbf{p}_i^{n_k+1})_k$ have weak limits v_i^∞ and $\operatorname{div} \mathbf{p}_i^\infty$, respectively. Since $\mathbf{p}_1^{n_k+1} \in \arg\min_{\mathbf{p}_1 \in K_1} \|\operatorname{div} \mathbf{p}_1 + v_2^{n_k} + g\|_{L^2(\Omega)}^2$ and $\mathbf{p}_2^{n_k+1} \in \arg\min_{\mathbf{p}_2 \in K_2} \|\operatorname{div} \mathbf{p}_2 + v_1^{n_k} + g\|_{L^2(\Omega)}^2$ we get from (2.15)

$$\|\operatorname{div} \mathbf{p}_1^{n_k+1} + v_2^{n_k} + g\|_{L^2(\Omega)}^2 \leq \|v_1^{n_k} + v_2^{n_k} + g\|_{L^2(\Omega)}^2 \quad \text{and}$$
$$\|\operatorname{div} \mathbf{p}_2^{n_k+1} + v_1^{n_k} + g\|_{L^2(\Omega)}^2 \leq \|v_1^{n_k} + v_2^{n_k} + g\|_{L^2(\Omega)}^2.$$

Passing in (2.15) for the suitable subsequence $(n_k)_k$ to the limit, we obtain that

$$\|v_1^\infty + v_2^\infty + g\|_{L^2(\Omega)}^2 - \|\operatorname{div} \mathbf{p}_1^\infty + v_2^\infty + g\|_{L^2(\Omega)}^2 = 0 \quad \text{and}$$
$$\|v_1^\infty + v_2^\infty + g\|_{L^2(\Omega)}^2 - \|\operatorname{div} \mathbf{p}_2^\infty + v_1^\infty + g\|_{L^2(\Omega)}^2 = 0.$$

By the strict convexity of $\|\cdot\|_{L^2(\Omega)}^2$ and the optimality of $\mathbf{p}_i^\infty$ we get

$$v_i^\infty = \operatorname{div} \mathbf{p}_i^\infty$$

for $i = 1, 2$.

Step 4: We are left to show that the sequence $(u^n)_n$ generated by Algorithm 2.3 converges to a solution of (1.1). We recall, that a function $u^* \in L^2(\Omega)$ solves (1.1) if and only if there exists $\mathbf{p}^* \in H_0(\operatorname{div}, \Omega)$ with $|\mathbf{p}^*(x)|_{\ell^2} \leq \alpha$ f.a.a. $x \in \Omega$ such that
  (i) $\operatorname{div} \mathbf{p}^* + g = u^*$ and
  (ii) $\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^* + g), \mathbf{p} - \mathbf{p}^* \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0$ for all $\mathbf{p} \in H_0(\operatorname{div}, \Omega)$ with $|\mathbf{p}(x)|_{\ell^2} \leq \alpha$ f.a.a. $x \in \Omega$.
Since $(v_i^n)_n$ is bounded in $L^2(\Omega)$ we also have that $(u^n)_n$ is bounded in $L^2(\Omega)$. For all $n \geq 1$ we have from step 1 that $v_i^n = \operatorname{div} \tilde{\mathbf{v}}_i^n$, where $(\tilde{\mathbf{v}}_i^n)_n$ is bounded in $L^2(\Omega)$, because $\tilde{\mathbf{v}}_i^n \in K_i$. Setting $\tilde{\mathbf{v}}^n := \tilde{\mathbf{v}}_1^n + \tilde{\mathbf{v}}_2^n$ we can write

$$u^{n+1} = g + \operatorname{div} \tilde{\mathbf{v}}_1^{n+1} + \operatorname{div} \tilde{\mathbf{v}}_2^{n+1} = g + \operatorname{div} \tilde{\mathbf{v}}^{n+1}$$

for all $n \geq 1$. Then let us take a suitable subsequence such that $u^{n_k+1} \rightharpoonup u^\infty$ and $\mathbf{p}^{n_k+1} \rightharpoonup \mathbf{p}^\infty$ and passing to the limit yields

$$(2.17) \qquad\qquad\qquad\qquad u^\infty = g + \operatorname{div} \tilde{\mathbf{v}}^\infty.$$

By the inequality in (2.11) we have that

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_1^{n+1} + f_1^{n+1}), \mathbf{p}_1 - \mathbf{p}_1^{n+1} \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_2^{n+1} + f_2^{n+1}), \mathbf{p}_2 - \mathbf{p}_2^{n+1} \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2$$

for all $n \geq 0$, which is equivalent to

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_1^{n+1} + v_2^n + g), \mathbf{p}_1 - \mathbf{p}_1^{n+1} \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}_2^{n+1} + v_1^n + g), \mathbf{p}_2 - \mathbf{p}_2^{n+1} \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2.$$

Since all quantities are bounded, we take suitable subsequences n_k such that $\mathbf{p}_i^{n_k+1} \rightharpoonup \mathbf{p}_i^\infty$ for $i = \{1, 2\}$ and $v_i^{n_k} \rightharpoonup v_i^\infty = \operatorname{div} \mathbf{p}_i^\infty$ (see step 3) and we get

$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^\infty + g), \mathbf{p}_1 - \mathbf{p}_1^\infty \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_1 \in K_1$$
$$\langle (-\operatorname{div})^*(\operatorname{div} \mathbf{p}^\infty + g), \mathbf{p}_2 - \mathbf{p}_2^\infty \rangle_{H_0(\operatorname{div}, \Omega)' \times H_0(\operatorname{div}, \Omega)} \leq 0 \quad \forall \mathbf{p}_2 \in K_2.$$

Summing up the latter two inequalities yields

$$\langle(-\operatorname{div})^*(\operatorname{div}\mathbf{p}^\infty + g), \mathbf{p} - \mathbf{p}^\infty\rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0 \quad \forall \mathbf{p} := \mathbf{p}_1 + \mathbf{p}_2 \in K_1 + K_2,$$

which is, following the same arguments as in the proof of Theorem 2.8, equivalent to

$$\langle(-\operatorname{div})^*(\operatorname{div}\mathbf{p}^\infty + g), \mathbf{p} - \mathbf{p}^\infty\rangle_{H_0(\operatorname{div},\Omega)' \times H_0(\operatorname{div},\Omega)} \leq 0$$

for all $\mathbf{p} \in H_0(\operatorname{div}, \Omega)$ with $|\mathbf{p}|_{\ell^2} \leq \alpha$ a.e. on Ω. This together with (2.17) and the fact that $|\mathbf{p}^\infty|_{\ell^2} \leq \alpha$ a.e. on Ω shows that u^∞ is a minimizer of (1.1). By the uniqueness of the minimizer, any convergent subsequence of $(u^n)_n$ converges to u^∞ and hence $u^n \rightharpoonup u^\infty$ for $n \to \infty$.

From (2.15) we directly obtain $\|u^n\|_{L^2(\Omega)} \geq \|u^{n+1}\|_{L^2(\Omega)}$. This together with the weak convergence $u^n \rightharpoonup u^\infty$ implies the strong convergence. $\qquad\square$

Remark 2.13. Alternatively to the boundedness assumption on $(f_i^n)_n$, $i = 1, 2$, in Theorem 2.12 we may assume that $(v_i^n)_n$, $i = 1, 2$, is bounded in $L^2(\Omega)$ instead. This would obviously imply the boundedness of $(f_i^n)_n$ in $L^2(\Omega)$. However, we note again that the boundedness assumption on $(f_i^n)_n$, $i = 1, 2$, in Theorem 2.12 indeed holds in a finite dimensional setting, cf. Remark 2.9.

As for the sequential method, in the infinite dimensional case the boundedness of $(f_i^n)_n$ in $L^2(\Omega)$ is only ensured if $\|u_2^n - u_1^n\|_{L^2(\Omega)}$ converges to 0 sufficiently fast (for $n \to \infty$).

PROPOSITION 2.14. *Let* $(u_i^n)_n$, $i = 1, 2$ *be generated by Algorithm 2.3. If there exists a constant* $\gamma > 1$ *such that* $\|u_2^n - u_1^n\|_{L^2(\Omega)} \leq \frac{1}{n^\gamma}$ *for any* $n \geq 1$, *then* $(v_i^n)_n$ *and* $(f_i^n)_n$, $i = 1, 2$, *are bounded in* $L^2(\Omega)$.

Proof. By induction one shows that

$$v_1^n = \frac{1}{4}\sum_{i=1}^{n-1}(u_1^i - u_2^i) + \frac{1}{2}u_1^n - \frac{1}{2}g \quad \text{and} \quad v_2^n = \frac{1}{4}\sum_{i=1}^{n-1}(u_2^i - u_1^i) + \frac{1}{2}u_2^n - \frac{1}{2}g$$

for $n \geq 1$. By the triangle inequality and the boundedness of $(u_i^n)_n$ in $L^2(\Omega)$, the assertion follows.$\square$

2.5. Multi-subdomain. The domain decomposition methods presented in Algorithm 2.2 and Algorithm 2.3 can be naturally extended to a multi-domain splitting. Let $M \in \mathbb{N}$ be the number of overlapping subdomains and $(\theta_i)_{i=1}^M$ a partition of unity with the properties

(a') $\displaystyle\sum_{i=1}^M \theta_i \equiv 1$ and $\theta_i \geq 0$ a.e. on $\overline{\Omega}$ for $i = 1, 2, \ldots, M$,

(b') $\operatorname{supp}(\theta_i) \subset \overline{\Omega}_i$ for $i = 1, 2, \ldots, M$,

(c') $\theta_i \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\|\nabla\theta_i\|_{\mathbb{L}^\infty(\Omega)} < \infty$ for $i = 1, 2, \ldots, M$.

The multi-subdomain versions of Algorithm 2.2 and Algorithm 2.3 are stated in Algorithm 2.4 and Algorithm 2.5, respectively.

Algorithm 2.4 Alternating Multi-subdomain Version

Initialize: $u_i^0 (= 0) \in L^2(\Omega)$, $f_i^0 = 0 \in L^2(\Omega)$, $i = 1, ..., M$
for $n = 0, 1, 2, \ldots$ **do**
    for $i = 1, \ldots, M$ **do**
        $f_i^{n+1} = \sum_{j>i}(u_j^n - f_j^n) + \sum_{j<i}(u_j^{n+1} - f_j^{n+1}) + g$
        $u_i^{n+1} = \arg\min_{u_i \in L^2(\Omega)} \frac{1}{2}\|u_i - f_1^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_i |Du_i|$
    end for
    $u^{n+1} = g + \sum_{i=1}^M u_i^{n+1} - f_i^{n+1} (= u_M^{n+1})$
end for

Algorithm 2.5 Parallel Multi-subdomain Version

> **Initialize:** $v_i^0 = 0$ for $i = 1, 2, ..., M$
> **for** $n = 0, 1, 2, \ldots$ **do**
> $\quad f_i^{n+1} = \sum_{j \neq i} v_j^n + g, \ i = 1, ..., M$
> $\quad u_i^{n+1} = \arg\min_{u_i \in L^2(\Omega)} \frac{1}{2}\|u_i - f_i^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_i |Du_i|, \ i = 1, ..., M$
> $\quad v_i^{n+1} = \frac{(M-1)v_i^n + u_i^{n+1} - f_i^{n+1}}{M}, \ i = 1, ..., M$
> $\quad u^{n+1} = g + \sum_{i=1}^M v_i^{n+1}(= \frac{\sum_{i=1}^M u_i^{n+1}}{M})$
> **end for**

Similar to Lemma 2.7 and Theorem 2.12 we are able to prove that the sequence $(u^n)_n$, generated by Algorithm 2.4 or Algorithm 2.5, is bounded in $L^2(\Omega)$. Further, if $(f_i^n)_n$ for $i = 1, \ldots, M$ is bounded in $L^2(\Omega)$, then we can again show as in Theorem 2.8 and in Theorem 2.12 that $(u^n)_n$ converges strongly in $L^2(\Omega)$ to the minimizer of (1.1). We recall, that in a finite dimensional setting, the boundedness of $(f_i^n)_n$ is easily shown as above. In an infinite dimensional setting we only obtain the following result:

PROPOSITION 2.15. *Assume that $M \in \mathbb{N}$ is a fixed number.*
(a) *Let $(u_i^n)_n$, $i = 1, \ldots, M$ be generated by Algorithm 2.4. If there exists a constant $\gamma > 1$ such that $\|u_{i-1}^{n+1} - u_i^n\|_{L^2(\Omega)} \leq \frac{1}{n^\gamma}$ for any $n \geq 1$, then $(f_i^n)_n$ is bounded in $L^2(\Omega)$ for $i \in \{1, \ldots, M\}$, where we use the convention that $u_0^{n+1} := u_M^n$.*
(b) *Let $(u_i^n)_n$, $i = 1, \ldots, M$ be generated by Algorithm 2.5. If there exists a constant $\gamma > 1$ such that $\| \sum_{\substack{j \neq i}}^M (u_i^n - u_j^n)\|_{L^2(\Omega)} \leq \frac{1}{n^\gamma}$ for any $n \geq 1$ and $i \in \{1, \ldots, M\}$, then $(v_i^n)_n$ and $(f_i^n)_n$ are bounded in $L^2(\Omega)$.*

Proof. (a) By induction one shows that

$$f_i^n = \sum_{\ell=1}^{n-1} (u_{i-1}^{\ell+1} - u_i^\ell) + u_{i-1}^1, \qquad \text{for } i = 1, \ldots, M \quad \text{and } n \geq 1,$$

where $u_0^1 := g$. The assertion follows then by the triangle inequality.
(b) Again by induction one shows that

$$v_i^n = \frac{1}{M^2} \sum_{k=1}^{n-1} \sum_{\substack{j \neq i}}^M (u_i^k - u_j^k) + \frac{1}{M} u_i^n - \frac{1}{M} g, \qquad \text{for } i = 1, \ldots, M \quad \text{and } n \geq 1.$$

Since $(u_i^n)_n$ is bounded in $L^2(\Omega)$, we obtain the assertion. □

Moreover, we emphasize, that for the sequential version, see Algorithm 2.4, we obtain as in Lemma 2.7 the monotonicity $\|u_1^n\|_{L^2(\Omega)} \geq \|u_2^n\|_{L^2(\Omega)} \geq \ldots \geq \|u_M^n\|_{L^2(\Omega)} \geq \|u_1^{n+1}\|_{L^2(\Omega)}$ for $n \geq 1$. Consequently, for $i \in \{1, \ldots, M\}$ the sequence $(u_i^n)_n$ converges strongly to a minimizer of the global problem.

Remark 2.16. In the same manner as presented above we may construct domain decomposition methods for (1.1) with $\int_\Omega |Du|_r$ for any $1 \leq r < +\infty$. By using the equivalence of norms in finite dimensions, in particular, that there exist positive constants $\overline{C}_{r^*}, \underline{C}_{r^*}$ such that $\underline{C}_{r^*}|v|_{\ell^{r^*}} \leq |v|_{\ell^2} \leq \overline{C}_{r^*}|v|_{\ell^{r^*}}$, where $\frac{1}{r} + \frac{1}{r^*} = 1$, the convergence proofs follow the same lines as the proofs above.

Remark 2.17. Actually the proposed domain decomposition algorithms may be used to solve the more general problem

$$(2.18) \qquad \arg\min_{u \in L^2(\Omega)} \frac{1}{2}\|Tu - g\|_{L^2(\Omega)}^2 + \alpha \int_\Omega |Du|,$$

where $T : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator, not annihilating constant functions, in order to guarantee existence of a minimizer of (2.18) [1].

Then, by using an operator splitting technique [16], a solution u^* of (2.18) for any given $u^0 \in L^2(\Omega)$ is obtained by iteratively solving

$$(2.19) \qquad u^{k+1} = \underset{u \in L^2(\Omega)}{\arg \min} \|u - (u^k + \frac{1}{\gamma} T^*(g - Tu^k))\|_{L^2(\Omega)}^2 + \alpha \int_\Omega |Du|, \qquad \text{for } k \geq 0,$$

where $\gamma > \|T\|^2$, see for example [17, 18]. Since the minimization problem in (2.19) is of the form (1.1), i.e., a denoising type of problem, the proposed domain decomposition methods may be applied to solve it.

3. Subspace minimization. Let us consider, for example, the subspace minimization with respect to u_1, i.e.,

$$(3.1) \qquad u_1^{n+1} = \underset{u_1 \in L^2(\Omega)}{\arg \min} \frac{1}{2}\|u_1 - f_1^{n+1}\|_{L^2(\Omega)}^2 + \alpha \int_\Omega \theta_1 |Du_1|.$$

We present two different approaches on how to compute the solution of (3.1) by solving a minimization on Ω_1 only. The first approach first discretizes (3.1) and then restricts the optimization process to the subdomain Ω_1, while the second approach restricts the minimization problem in an infinite dimensional setting before discretization. These two approaches lead to nearly equal but still different discrete subspace problems.

3.1. Discretize before restriction. We start by introducing a discretization of the subproblems constituted by the above introduced domain decomposition methods. Therefore, let Ω^h be a discrete rectangular image domain containing $N_1 \times N_2$ pixels, $N_1, N_2 \in \mathbb{N}$. We approximate functions u by discrete functions, denoted by u^h. The considered function spaces are $X = \mathbb{R}^{N_1 \times N_2}$ and $Y = X \times X$. For $u^h \in X$ and $\mathbf{p}^h = (p^{h,1}, p^{h,2}) \in Y$ we use the norms $\|u^h\|_X := (\sum_{x \in \Omega^h} |u^h(x)|^2)^{1/2}$ and $\|\mathbf{p}^h\|_Y^2 := \|p^{h,1}\|_X^2 + \|p^{h,2}\|_X^2$. On Ω^h the discrete gradient $\nabla_\Omega^h : X \to Y$ and the discrete divergence $\mathrm{div}_\Omega^h : Y \to X$ are defined in a standard way by forward and backward differences such that $\mathrm{div}_\Omega^h = -(\nabla_\Omega^h)^*$; see for example [34].

The discretized version of (3.1) using the above notation and definitions is written as

$$(3.2) \qquad u_1^{h,n+1} = \underset{u_1^h \in X}{\arg \min} \frac{1}{2}\|u_1^h - f_1^{h,n+1}\|_X^2 + \alpha \sum_{x \in \Omega^h} \theta_1^h(x)|\nabla_\Omega^h u_1^h(x)|_{\ell^2}.$$

Let Ω^h be decomposed into overlapping subdomains Ω_i^h, $i = 1, \ldots, M$ such that $\Omega^h = \bigcup_{i=1}^M \Omega_i^h$ and for any $i \in \{1, \ldots, M\}$ there exists at least one $j \in \{1, \ldots, M\} \setminus \{i\}$ such that $\Omega_i^h \cap \Omega_j^h \neq \emptyset$. Due to this splitting, we define $X_1 := \mathbb{R}^{|\Omega_1^h|}$ and $Y_1 = X_1 \times X_1$, and accordingly the norms $\|u^h\|_{X_1} := (\sum_{x \in \Omega_1^h} |u^h(x)|^2)^{1/2}$, $\|\mathbf{p}^h\|_{Y_1}^2 := \|p^{h,1}\|_{X_1}^2 + \|p^{h,2}\|_{X_1}^2$ for $u^h \in X_1$ and $\mathbf{p}^h \in Y_1$. Let $u^h \in X$, then by $u^h_{|\Omega_1^h}$ we define the restriction of u^h to Ω_1^h and consequently $u^h_{|\Omega_1^h} \in X_1$. Since $\theta_1^h(x) = 0$ for all $x \in \Omega^h \setminus \Omega_1^h$ we can write the above minimization problem as

$$(3.3) \qquad u_1^{h,n+1} = \begin{cases} f_1^{h,n+1} & \text{in } \Omega^h \setminus \Omega_1^h \\ \underset{u_1^h{}_{|\Omega_1^h} \in X_1}{\arg \min} \frac{1}{2}\|u_1^h - f_1^{h,n+1}\|_{X_1}^2 + \alpha \sum_{x \in \Omega_1^h} \theta_1^h(x)|\nabla_\Omega^h u_1^h(x)|_{\ell^2} & \text{in } \Omega_1^h, \end{cases}$$

where $u_1^h \in X$ is such that $u_1^h(x) = f_1^{h,n+1}(x)$ for $x \in \Omega^h \setminus \Omega_1^h$. Hence, in order to obtain $u_1^{h,n+1}$ only a minimization problem in Ω_1^h has to be solved.

Let us now discuss how we algorithmically solve the optimization problem in (3.3). Due to the presence of the function θ_1, respectively θ_1^h, no standard total variation minimization technique can be used. Instead one may need to adapt one of these standard approaches to this more general functional. We note, that the minimization of locally weighted total variation have been

already considered in the literature and an algorithm to solve a minimization problem of the type as in (3.3) is already presented in [39]. In particular, in [39] the primal-dual algorithm of [10] is adapted to a locally weighted total variation regularization. This method requires, that the locally distributed weights are strictly positive, i.e., $\theta_1^h(x) > 0$ for every $x \in \Omega_1^h$ in (3.3). The proposed domain decomposition algorithms and their theory require that $\text{supp}(\theta_1) \subseteq \Omega_1^h$. Hence, in our discrete setting one may easily set $\theta_1^h > 0$ in Ω_1^h and $\theta_1^h = 0$ in $\Omega^h \setminus \Omega_1^h$, which allows to use the algorithm provided in [39]. Nevertheless, we derive an algorithm for locally weighted total variation, which does not require the strict positivity, and hence may be used to solve (3.1) and (3.2) as well. Actually we adapt the split Bregman algorithm [30], which we are explaining next.

We first replace $(\nabla_\Omega^h u_1^h)_{|_{\Omega_1^h}}$ with $\mathbf{d}_1^h$ and then enforce the constraint $(\nabla^h u_1^h)_{|_{\Omega_1^h}} = \mathbf{d}_1^h$ by applying the Bregman iteration yielding

(3.4)
$$(u_{1,\Omega_1^h}^{h,k+1}, \mathbf{d}_1^{h,k+1}) \in \underset{u_{1|_{\Omega_1^h}}^h, \mathbf{d}_1^h}{\arg\min} \frac{1}{2}\|u_1^h - f_1^{h,n+1}\|_{X_1}^2 + \alpha \sum_{x \in \Omega_1^h} \theta_1^h |\mathbf{d}_1^h|_{\ell^2} + \frac{\mu}{2}\|\mathbf{d}_1^h - (\nabla_\Omega^h u_1^h)_{|_{\Omega_1^h}} - \mathbf{b}_1^k\|_{Y_1}^2$$
$$\mathbf{b}_1^{k+1} = \mathbf{b}_1^k + (\nabla_\Omega^h u_1^{h,k+1})_{|_{\Omega_1^h}} - \mathbf{d}_1^{h,k+1}$$

where $\mu > 0$ is a Lagrange multiplier and $\mathbf{b}_1^k \in Y_1$. The obtained minimization problem in (3.4) is then iteratively minimized, first with respect to $u_{1|_{\Omega_1^h}}^h$ and then to $\mathbf{d}_1^h$. Note, that ∇_Ω^h is a quite local operator, i.e., it affects only neighbouring pixels. Hence, by carefully considering the restriction to Ω_1^h (i.e., we use Dirichlet boundary conditions on the interface between Ω_1^h and $\Omega^h \setminus \Omega_1^h$), $u_{1,\Omega_1^h}^{h,k+1} \in X_1$ is obtained, as in [30], by solving a linear system only of size $|\Omega_1^h|$, see Appendix A. The optimal value $\mathbf{d}_1^{h,k+1}$ can be computed by a shrinkage formula [56], i.e.,

$$\mathbf{d}_1^{h,k+1} = \max\left\{|(\nabla_\Omega^h u_1^{h,k+1})_{|_{\Omega_1^h}} + \mathbf{b}_1^k|_{\ell^2} - \frac{\alpha}{\mu}\theta_1^h, 0\right\} \frac{(\nabla_\Omega^h u_1^{h,k+1})_{|_{\Omega_1^h}} + \mathbf{b}_1^k}{|(\nabla_\Omega^h u_1^{h,k+1})_{|_{\Omega_1^h}} + \mathbf{b}_1^k|_{\ell^2}},$$

where we follow the convention that $0 \cdot \frac{0}{0} = 0$.

Since this adapted split Bregman method does not require the strict positivity of θ_1 and θ_1^h, it is also able to solve (3.1) and (3.2) by just adjusting the respective quantities accordingly (i.e., without restricting to Ω_1^h).

Remark 3.1 (Overlapping to nonoverlapping). In a discrete setting the continuity assumption on θ_i^h, for $i = 1, \ldots, M$, is obsolete. Hence we may let the overlapping-size go to 0 yielding a nonoverlapping decomposition. That is

$$\theta_i^h(x) = \begin{cases} 1 & \text{if } x \in \Omega_i^h \\ 0 & \text{else} \end{cases}$$

for $i = 1, \ldots, M$. Then the subspace minimization problems read as

$$\underset{u_i \in X_i}{\arg\min} \frac{1}{2}\|u_i^h - f_i^{h,n+1}\|_{X_i}^2 + \alpha \sum_{x \in \Omega_i^h} |\nabla_\Omega^h u_i^h(x)|_{\ell^2},$$

$i = 1, \ldots, M$. Thus in a discrete setting, using this discretization and restriction approach, in the limit case of a nonoverlapping decomposition Algorithm 2.2 and Algorithm 2.3 become the Block Gauss-Seidel and Relaxed Block Jacobi method of [45], respectively.

3.2. Restrict before discretization. Let us turn back to the infinite dimensional subdomain problem (3.1). Since the partition of unity is such that $\text{supp}(\theta_1) \subseteq \Omega_1^h$, we have $\int_\Omega \theta_1 |Du_1| = \int_{\Omega_1} \theta_1 |Du_1|$, cf. Lemma 2.2. Hence, by the optimality of u_1^{n+1} we get $f_1^{n+1} - u_1^{n+1} \in \partial\alpha \int_{\Omega_1} \theta_1 |Du_1^{n+1}|$.

That is

$$(f_1^{n+1} - u_1^{n+1}, v - u_1^{n+1}) + \alpha \int_{\Omega_1} \theta_1 |Du_1^{n+1}| \le \alpha \int_{\Omega_1} \theta_1 |Dv| \qquad \forall v \in L^2(\Omega).$$

This inequality holds if

$$\int_{\Omega \setminus \Omega_1} (f_1^{n+1} - u_1^{n+1})(v - u_1^{n+1})\mathrm{d}x \le 0 \qquad \text{and}$$

$$\int_{\Omega_1} (f_1^{n+1} - u_1^{n+1})(v - u_1^{n+1})\mathrm{d}x + \alpha \int_{\Omega_1} \theta_1 |Du_1^{n+1}| \le \alpha \int_{\Omega_1} \theta_1 |Dv|$$

for all $v \in L^2(\Omega)$. Hence u_1^{n+1} fulfilling these two latter inequalities is a minimizer of the subspace minimization problem (3.1). By the uniqueness of the minimizer, see Proposition 2.3, we therefore obtain that

$$(3.5) \qquad u_1^{n+1} = \begin{cases} f_1^{n+1} & \text{in } \Omega \setminus \Omega_1 \\ \underset{u_1 \in L^2(\Omega_1)}{\arg\min} \ \frac{1}{2}\|u_1 - f_1^{n+1}\|_{L^2(\Omega_1)}^2 + \alpha \int_{\Omega_1} \theta_1 |Du_1| & \text{in } \Omega_1. \end{cases}$$

Thus, generating the minimizer of (3.1) results in solving an optimization problem in Ω_1 only. As above, due to the presence of the function θ_1, usual total variation minimization techniques cannot be used. For solving the optimization problem in (3.5), we want to use again the split Bregman method adapted to locally weighted total variation. Therefore, we discretize (3.5) by using the above notations and definitions yielding

$$(3.6) \quad u_1^{h,n+1} = \begin{cases} f_1^{h,n+1} & \text{in } \Omega_2^h \setminus \Omega_1^h \\ \underset{u_1^h{}_{|\Omega_1^h} \in X_1}{\arg\min} \ \frac{1}{2}\|u_1^h{}_{|\Omega_1^h} - f_1^{h,n+1}\|_{X_1}^2 + \alpha \sum_{x \in \Omega_1^h} \theta_1^h(x) |\nabla_{\Omega_1}^h(u_1^h{}_{|\Omega_1^h}(x))|_{\ell^2} & \text{in } \Omega_1^h, \end{cases}$$

where $\nabla_{\Omega_1}^h$ denotes the standard gradient on Ω_1^h with zero Neumann boundary conditions on $\partial\Omega_1$, cf. Section 3.1. Similar as above, one derives the adapted split Bregman algorithm, presented in Algorithm 3.1, where $\Delta_{\Omega_1}^h$ denotes the respective standard discrete Laplace operator on Ω_1^h.

Algorithm 3.1 Split Bregman

Initialize: $\mathbf{d}_1^{h,0} = 0 = \mathbf{b}_1^0$ and $k = 0$
while stopping criterion does not hold **do**
    Solve $(I - \mu\Delta_{\Omega_1}^h)u_1^{h,k+1}{}_{|\Omega_1^h} = f_1^{h,n+1} - \mu \operatorname{div}_{\Omega_1}^h(\mathbf{d}_1^{h,k} - \mathbf{b}_1^k)$

    $\mathbf{d}_1^{h,k+1} = \max\{|\nabla_{\Omega_1}^h(u_1^{h,k+1}{}_{|\Omega_1^h}) + \mathbf{b}_1^k|_{\ell^2} - \frac{\alpha}{\mu}\theta_1^h, 0\}\dfrac{\nabla_{\Omega_1}^h(u_1^{h,k+1}{}_{|\Omega_1^h}) + \mathbf{b}_1^k}{|\nabla_{\Omega_1}^h(u_1^{h,k+1}{}_{|\Omega_1^h}) + \mathbf{b}_1^k|_{\ell^2}}$

    $\mathbf{b}_1^{k+1} = \mathbf{b}_1^k + \nabla_{\Omega_1}^h(u_1^{h,k+1}{}_{|\Omega_1^h}) - \mathbf{d}_1^{h,k+1}$

    $k = k + 1$
end while

Let us mention that all the results presented in Section 3.1 and Section 3.2 hold symmetrically for the minimization with respect to u_i, $i = 2, \dots, M$, and that the notations should be just adjusted accordingly.

3.3. Comparison of the two restriction approaches. If we compare (3.3) with (3.6), we observe, that the two above discussed restriction and discretization approaches lead to quite similar optimization problems, while the difference lies in the different discrete gradient operators. This difference indeed has a significant impact. While the "discretize before restriction" approach

(DbR) allows for a nonoverlapping domain decomposition, see Remark 3.1, the "restrict before discretization" approach (RbD) does not support this situation. This is due to the fact, that the restriction process in the latter approach is done in the continuous setting, where the continuity of θ_i is necessary. Moreover, from (3.6) we see that in case of a nonoverlapping decomposition, due to the "local" operator $\nabla^h_{\Omega_i}$, no information from outside of Ω_i is entering Ω_i, which would result in a wrong behaviour, see Figure 1. In particular in Figure 1 we contrast the reconstructions of the two approaches for a nonoverlapping decomposition. The figures show, as expected, that the "restrict before discretization" approach creates an artificial edge at the interface of the subdomains, while the "discretize before restriction" approach still works correct.
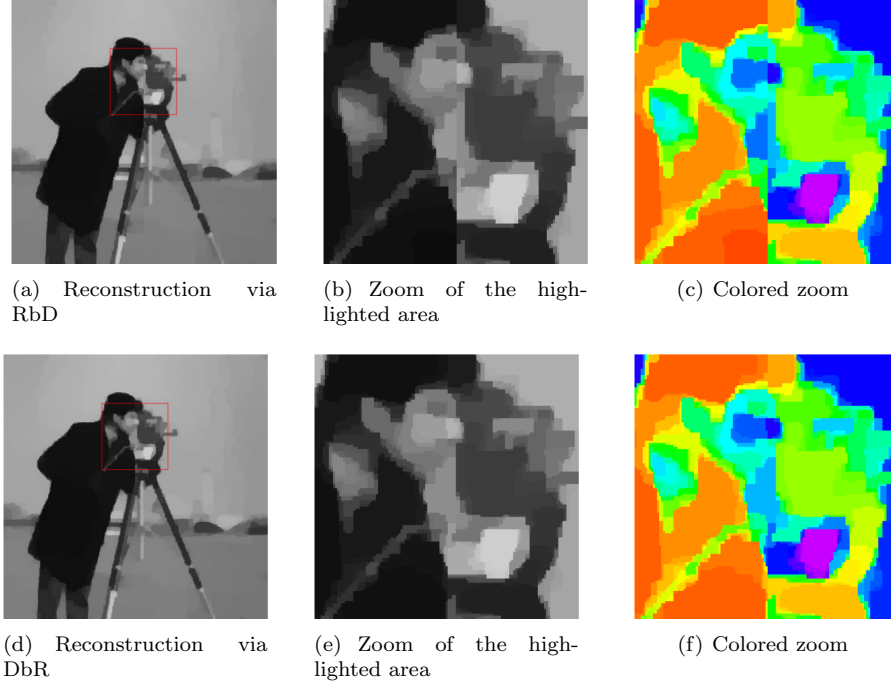

(a) Reconstruction via RbD

(b) Zoom of the high-lighted area

(c) Colored zoom

(d) Reconstruction via DbR

(e) Zoom of the high-lighted area

(f) Colored zoom

FIG. 1. *Reconstruction of an image of size 512×512 pixels corrupted by additive Gaussian white noise with $\sigma = 0.3$ using Algorithm 2.2 and a nonoverlapping splitting into 2 domains. In the first row we show the result via the "restrict before discretization" approach (RbD), while the second row shows the result via the "discretize before restriction" approach (DbR). In (b) and (e) we zoomed in on the in (a) and (d) highlighted area. In order to visualize the difference of the reconstructions, we color in (c) and (f) the zoomed area.*

However, in case of an overlapping decomposition, for which the proposed algorithms are actually constructed, both approaches seem to generate a very similar reconstruction, see Figure 2 where the overlapping size is 512×20 pixels. Moreover, performing several experiments, for both approaches the overall behaviour seems to be very similar, see Figure 3 for the example in Figure 2. More precisely, in all our experiments we observe that the sequential domain decomposition algorithm using the "restrict before discretization" approach always needs at most the same amount of iterations as the other approach, while in most of the experiments the algorithm even terminates one iteration earlier when using the "restrict before discretization" approach. In particular in Figure 3 we see, that with the "restrict before discretization" approach only 17 iterations till termination are needed while with the "discretize before restriction" approach 18 iterations are needed. In our numerical experiments in Section 4 we exclusively use the "first restrict and then discretize" approach, which seems theoretically more consistent, is finally even slightly easier to implement, since on the subdomains no specific Dirichlet boundary conditions have to be considered, and seems to be slightly faster (or at least not slower) than the other approach.
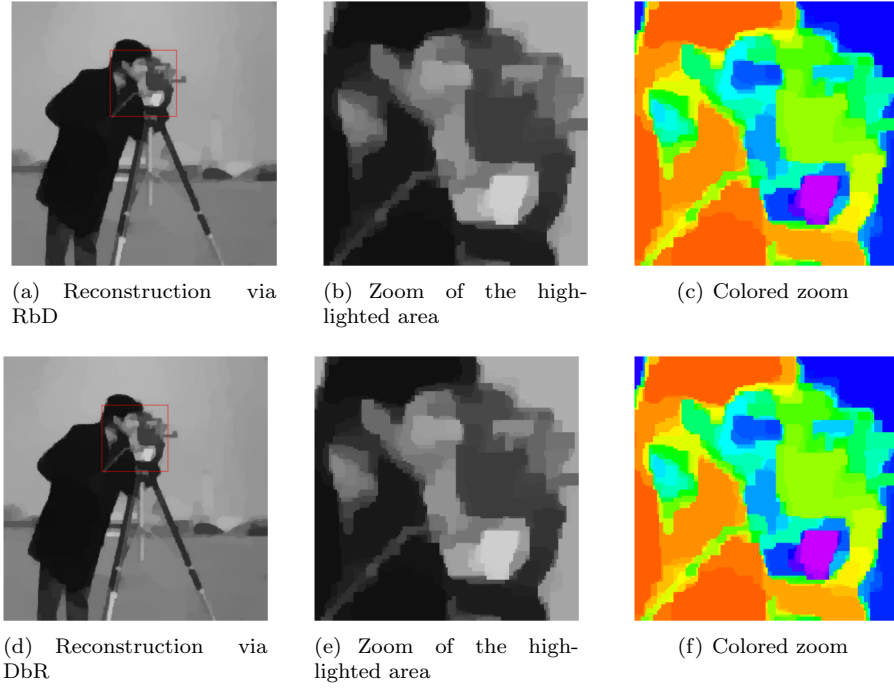
(a) Reconstruction via RbD

(b) Zoom of the highlighted area

(c) Colored zoom

(d) Reconstruction via DbR

(e) Zoom of the highlighted area

(f) Colored zoom

FIG. 2. *Reconstruction of an image of size 512×512 pixels corrupted by additive Gaussian white noise with $\sigma = 0.3$ using Algorithm 2.2 and an overlapping splitting into 2 domains. In the first row we show the result via the "restrict before discretization" approach (RbD), while the second row shows the result via the "discretize before restriction" approach (DbR). In (b) and (e) we zoomed in on the in (a) and (d) highlighted area. In order to visualize possible differences of the reconstructions, we color in (c) and (f) the zoomed area.*

(a) Convergence order

(b) $(\|u^n\|)_n$
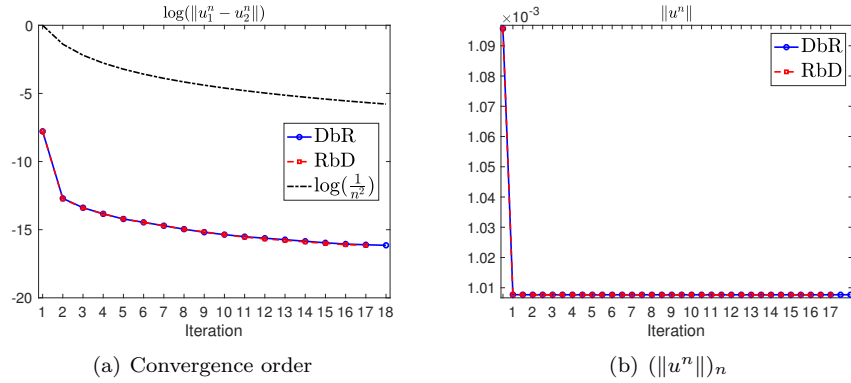
FIG. 3. *Performance of Algorithm 2.2 using the "restrict before discretization" approach (RbD) and "discretize before restriction" approach (DbR).*

3.4. Implementation issues. We recall that in each iteration of the proposed parallel domain decomposition algorithm we set

$$v_i^{n+1} = \frac{(M-1)v_i^n + u_i^{n+1} - f_i^{n+1}}{M}$$

for $i = 1, ..., M$, see Algorithm 2.5. Note, that by (3.5) we have, $u_i^{n+1} = f_i^{n+1}$ in $\Omega \setminus \Omega_i$. Since $v_i^0 = 0$, we iteratively obtain that supp $v_i^n \subset \Omega_i$ for all $n \in \mathbb{N}$. Thus, in our implementation we need to update v_i^n in Ω_i only. Consequently only u_i^{n+1} in Ω_i is needed, which is obtained by

solving an optimization problem restricted so Ω_i, see (3.5). Moreover, since $u^{n+1} = \frac{\sum_{i=1}^{M} u_i^{n+1}}{M} = g + \sum_{i=1}^{M} v_i^{n+1}$, we do not even need u_i^{n+1} for updating u^{n+1} and hence u_i^{n+1} does not have to be communicated, in case of a multi-processor implementation. Hence, in each subdomain we only need (to compute) quantities which are restricted to this subdomain, and the update $u_i^{n+1} = f_i^{n+1}$ in $\Omega \setminus \Omega_i$ is not performed at all.

For the alternating domain decomposition algorithm we obtain the same. In particular, since $u_i^n - f_i^n = 0$ in $\Omega \setminus \Omega_i$, we conclude that

$$u^n = g + \sum_{i=1}^{M} (u_{i\,|\Omega_i}^n - f_{i\,|\Omega_i}^n).$$

Note, that the same argumentation also holds in a discrete setting due to (3.3) and (3.6).

In order to speed up the minimization procedure in the subdomains, we only initialize in the first outer iteration the split Bregman algorithm as described in Algorithm 3.1. Later we use the approximation of the previous iterate. More precisely, let $\tilde{u}_1^{h,n}$ be the solution of the split Bregman iteration in the n-th outer iteration and $\tilde{\mathbf{d}}_1^{h,n}$, $\tilde{\mathbf{b}}_1^n$ the associated variables. Then in the next iteration we initialize the split Bregman iteration by $\mathbf{d}_1^{h,0} = \tilde{\mathbf{d}}_1^{h,n}$, and $\mathbf{b}_1^0 = \tilde{\mathbf{b}}_1^n$.

4. Experiments. In the following section we present numerical experiments for the proposed sequential and parallel algorithms for image denoising. The value of the parameter α is chosen arbitrarily and is not optimized in any way. For automatically choosing the regularization parameter α in (1.1) we refer the reader to [39] and references therein. As subdomain solver we use Algorithm 3.1 with $\mu = \frac{3}{\alpha}$, which is terminated as soon as

$$(4.1) \qquad \frac{\|u_i^{h,k} - u_i^{h,k-1}\|_{\ell^1}}{\|u_i^{h,k}\|_{\ell^1}} \leq \text{tol}$$

holds in subdomain $i \in \{1, \ldots, M\}$ for the first time, where tol is a predefined tolerance.

All the computations presented are done in Matlab on a MacBook Pro with 2.5 GHz Intel Core i7 processor (possesses 4 cores).

4.1. Domain decomposition. For simplicity let $\Omega \subset \mathbb{R}^2$ be a rectangular domain $[a, b] \times [c, d]$, with $a < b$ and $c < d$. Then we decompose Ω into $M \in \mathbb{N}$ subdomains Ω_i such that

$$\Omega_i = [a_i, b_i] \times [c, d] \quad \text{for } i = 1, \ldots, M$$

where $a =: a_1 < a_2 < b_1 < a_3 < b_2 < \ldots < b_{M-1} < b_M := b$, see Figure 4 for $M = 3$. The overlapping width is then $w_i = |b_i - a_{i+1}|$.
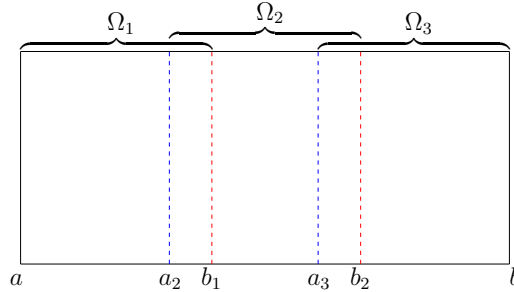

FIG. 4. *Overlapping domain decomposition of Ω into Ω_1, Ω_2 and Ω_3.*
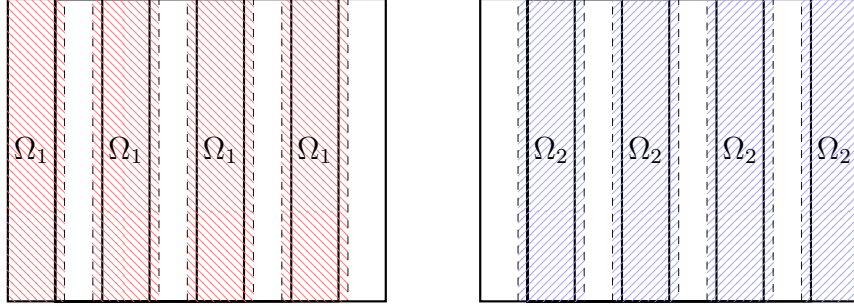
FIG. 5. *Partitioning of an overlapping decomposition by colouring technique.*

The auxiliary function θ_i, $i = 1, \ldots, M$, is chosen as

$$\theta_i(x) = \begin{cases} 1 & \text{if } x \in \Omega_i \setminus (\Omega_{i-1} \cup \Omega_{i+1}) \\ \frac{b_i - x}{b_i - a_{i+1}} & \text{if } x \in \Omega_i \cap \Omega_{i+1} \\ \frac{x - a_i}{b_{i-1} - a_i} & \text{if } x \in \Omega_i \cap \Omega_{i-1} \\ 0 & \text{else (if } x \in \Omega \setminus \Omega_i) \end{cases} \qquad \text{for } i = 1, \ldots, M,$$

where $\Omega_0 = \Omega_{M+1} = \emptyset$. Compare with Figure 6 and Figure 7 for a decomposition into 2 and 3 domains, respectively.

In the parallel multi-subdomain method (Algorithm 2.5) for any $n \geq 0$ the value v_i^{n+1} is obtained by a weighted sum of the previous iterate v_i^n and $u_i^{n+1} - f_i^{n+1}$, whereas the latter term is weighted by $\frac{1}{M}$. This weight tends to zero as the number of subdomains M grows, which may lead to a crucial decrease of the convergence speed of the algorithm. In order to overcome this behaviour, we may use a so-called *colouring technique*; see e.g. [53]. That is, the image domain Ω is partitioned into a fixed number M_c of classes of overlapping subdomains, whereby each class is coloured by a different colour, i.e.,

$$\Omega = \bigcup_{j=1}^{M_c} \Omega_j,$$

where Ω_j is the union of disjoint subdomains with the same colour. An example of an overlapping decomposition of a rectangular domain Ω into 8 subdomains coloured by 2 different colours is depicted in Figure 5. We note, that in general the disjoint domains with same colour cannot be solved in parallel without introducing additional new constraints, as the following example, borrowed from Warga [57], shows.

EXAMPLE 4.1. *Let* $V := [0,1] \times \{0\} \times [0,1]$, $V_1 := \{(c,0,0) \; c \in [0,1]\}$, $V_3 := \{(0,0,c) \; c \in [0,1]\}$, *and* $\varphi : V \to \mathbb{R}$ *given by* $\varphi(x) = |x_1 - x_3| - \min\{x_1, x_3\}$, *where* $x = (x_1, x_2, x_3)$. *We have that* $\mathbf{0} = \arg\min_{x_i \in V_i} \varphi(x)$ *for* $i \in \{1,3\}$, *while* $(1,0,1) = \arg\min_{x \in V} \varphi(x)$.

However, if the problem is additively separable with respect to the considered disjoint decomposition, then the problem can be solved easily in parallel on the disjoint domains. Since this property holds for our considered subdomain problems with respect to the disjoint domains with same colour, a partitioning with colouring technique changes the update of v_i^{n+1} to

$$v_i^{n+1} = \frac{(M_c - 1)v_i^n + u_i^{n+1} - f_i^{n+1}}{M_c}, \qquad \text{for } n \geq 0,$$

where $M_c \leq M$. In particular if M is large, then $M_c \ll M$. For instance, decomposing and colouring as in Figure 5, keeps $M_c = 2$ for any number of subdomains $M > 1$.

4.2. Sequential algorithm. The scope of this section is to illustrate by simple examples the main properties of the algorithms, as proven in our theoretical analysis. In particular, we investigate the convergence order of $\|u_{i-1}^{n+1} - u_i^n\|_{L^2(\Omega)}$ for $i = 1, \ldots, M$ and we demonstrate the

monotonicity (2.10). Moreover, we emphasize the robustness in correct computing minimizers independently of the size of overlapping regions and of the number of subdomains. As tolerance for the subspace solver we set tol $= 10^{-6}$.

In Figure 6 we show an image corrupted by additive Gaussian noise with standard deviation $\sigma = 0.3$ and zero mean. This image is then reconstructed by means of total variation minimization. That is, the reconstruction is the solution of (1.1) with $\alpha = 0.5$. In order to obtain a minimizer of (1.1) we utilize the proposed alternating domain decomposition method, see Algorithm 2.2 and Algorithm 2.4. We test our algorithms for different numbers of subdomains, in particular we present results for $M \in \{2, 3, 6\}$, whereas the domain is split vertically, as shown in Figure 6 and Figure 7 for an overlapping decomposition into 2 and 3 domains respectively. There the red and blue lines indicate a right and left (inner) boundary of a subdomain respectively. The domain decomposition algorithm is terminated as soon as

$$\|u_M^n - u_{M-1}^n\|_{L^2(\Omega)} \le 10^{-7}$$

for the first time, which indicates that nearly no changes are to be expected. This stopping criterion makes sense, since by Theorem 2.8 and Corollary 2.10, together with their extension to the multi-domain case, any converging subsequence of $(u_i^n)_n$, $i = 1, \ldots, M$, has the same limit, minimizing (1.1). The reconstructions of Figure 6(a) and Figure 7(a) obtained via the proposed overlapping domain decomposition algorithm, see Algorithm 2.4, where the overlapping regions are of size 256×80 pixels, are shown in Figure 6(b) and Figure 7(b) respectively.
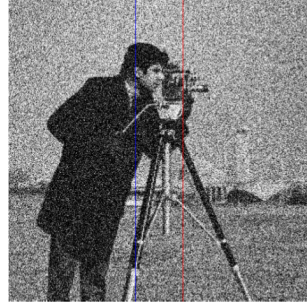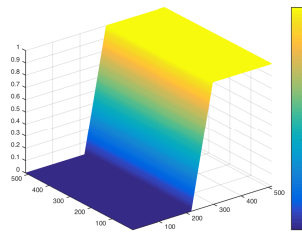

(a) Noisy image decomposed into 2 subdomains.

(b) Reconstruction.

(c) θ_1.
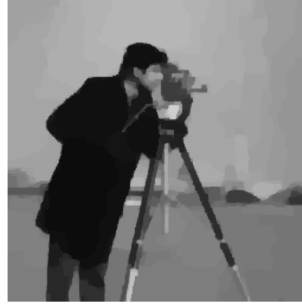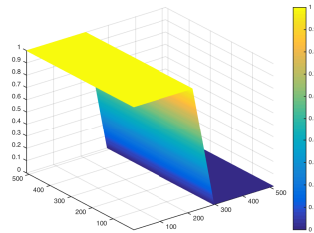
(d) θ_2.

Fig. 6. *Reconstruction of an image (512×512 pixels) corrupted by Gaussian white noise with standard deviation $\sigma = 0.3$ using the regularization parameter $\alpha = 0.5$. The noisy image is decomposed into 2 overlapping subdomains with overlap-size 512×80 pixels. The blue and red line indicate the interfaces of the overlapping region.*

For a decomposition into $M = 2$ subdomains, we depict in Figure 8(a) for different sizes of overlapping regions the decay of $(\|u_2^n - u_1^n\|_{L^2(\Omega)})_n$. In particular, we test for overlapping regions of size 256×20 pixels, 256×80 pixels, and 256×120 pixels. We detect, that for $\gamma = \frac{3}{2}$ we have $\|u_2^n - u_1^n\|_{L^2(\Omega)} \le \frac{1}{n^\gamma}$ for all n. This not only gives us an idea of the convergence order of the sequential domain decomposition algorithm, but also justifies the boundedness of $(f_i^n)_n$,
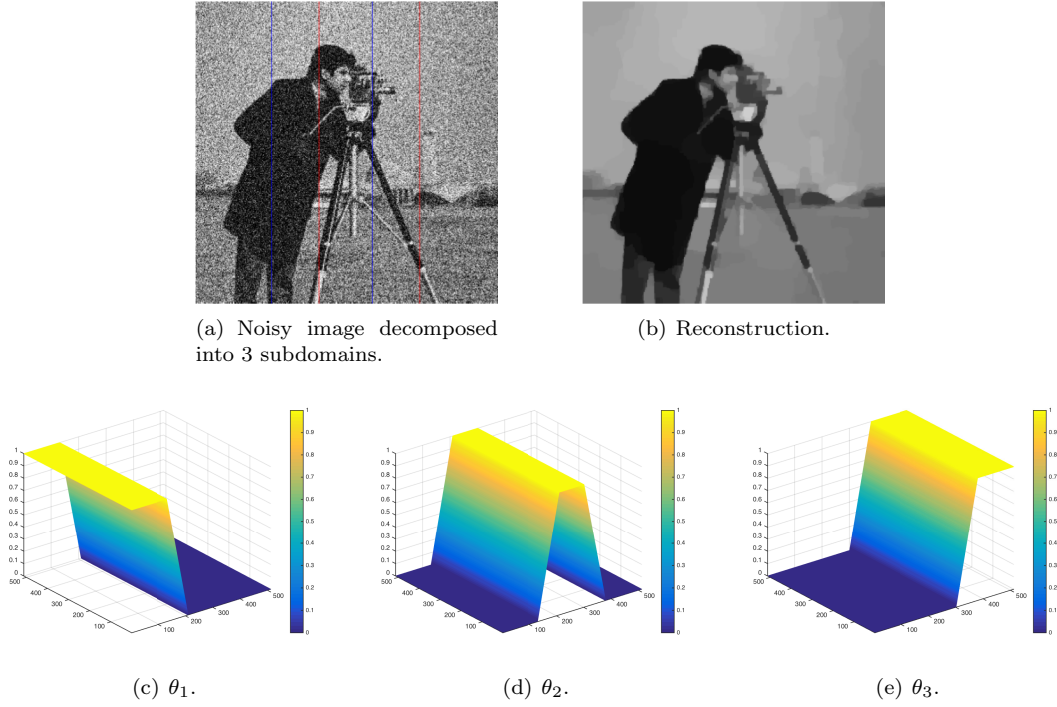
(a) Noisy image decomposed into 3 subdomains.

(b) Reconstruction.

(c) θ_1.

(d) θ_2.

(e) θ_3.

FIG. 7. *Reconstruction of an image (512×512 pixels) corrupted by Gaussian white noise with standard deviation $\sigma = 0.3$ using the regularization parameter $\alpha = 0.5$. The noisy image is decomposed into 3 overlapping subdomains with overlap-size 512×80 pixels. The blue and red line indicate the interfaces of the overlapping region.*

which is actually anyway theoretically ensured in a discrete setting, cf. Remark 2.9. Moreover, we observe that the larger the overlapping region the more iterations are needed till termination. The monotonicity (2.10) is depicted in Figure 8(b), where we use the notation $u^{n-1/2} := u_1^n$ and $u^n := u_2^n$ for all $n \in \mathbb{N}$. We observe, that after one outer iteration $\|u^n\|_{L^2(\Omega)}$ (where $n = 1, 1 + \frac{1}{2}, 2, \dots$) does not change visibly anymore.
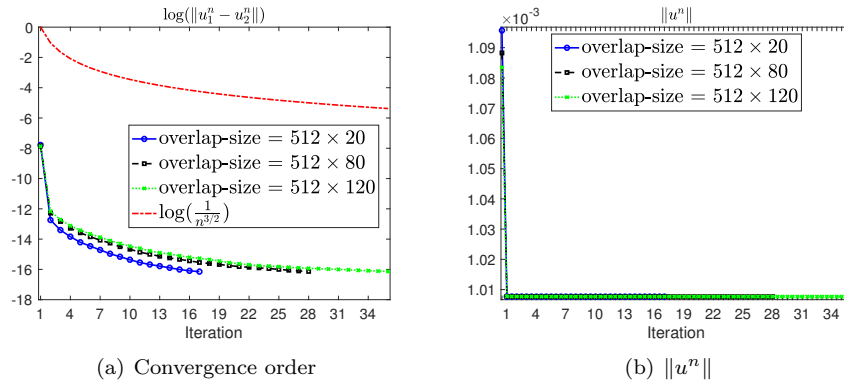

(a) Convergence order

(b) $\|u^n\|$

FIG. 8. *Performance record of the reconstruction of the image in Figure 6 with different overlapping-sizes.*

For $M = 3$ and $M = 6$, we depict in Figure 9 the performance of the sequential domain decomposition algorithm for overlapping sizes of 512×40 pixels. In Figure 9(a) and Figure 9(b) we present the decay of the norms $\|u_{i-1}^{n+1} - u_i^n\|_{L^2(\Omega)}$, $i = 1, \dots, M$, with respect to the outer

iteration n. As for the case $M = 2$, we observe, that $\|u_{i-1}^{n+1} - u_i^n\|_{L^2(\Omega)} \le \frac{1}{n^{3/2}}$ for $i = 1, \dots, M$, showing that the sequences $(f_i^n)_n$ are indeed bounded. Using the notation $u^{n+i/M} := u_i^{n+1}$ for $n = 0, 1, 2, \dots$ and $i = 1, \dots, M$, we depict in Figure 9(c) and 9(d) the monotonicity of $\|u^n\|_{L^2(\Omega)}$ with respect to the outer iterations n.
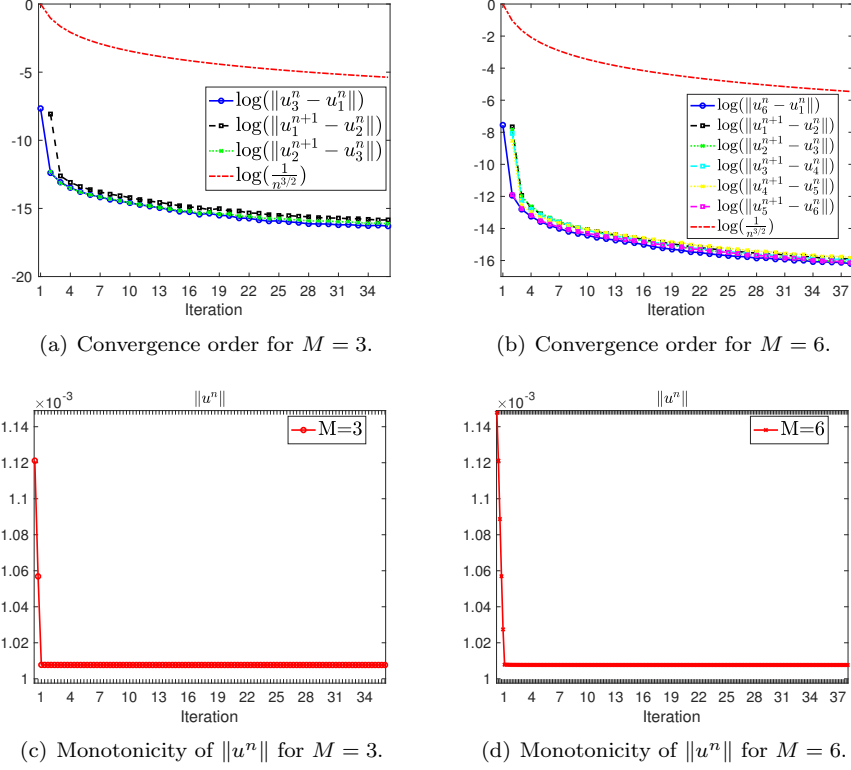

(a) Convergence order for $M = 3$.

(b) Convergence order for $M = 6$.

(c) Monotonicity of $\|u^n\|$ for $M = 3$.

(d) Monotonicity of $\|u^n\|$ for $M = 6$.

FIG. 9. *Performance record of the reconstruction of the image in Figure 6 with different numbers of subdomains.*

4.3. Parallel algorithm. Finally, we demonstrate the efficiency of the proposed parallel domain decomposition method (see Algorithm 2.5) when implemented on a multiple processor computer. We compare its performance with the split Bregman algorithm [30], which computes a solution of (1.1) without any domain decomposition. Notice that other algorithms for computing a solution of (1.1), see for example [6, 9], might be used as well, changing the result maybe qualitatively but not quantitatively. For a fair comparison, in the domain decomposition method we use the split Bregman method as subdomain solver, described in Algorithm 3.1, and stop it as soon as (4.1) holds for the first time for a given tolerance tol := tol(n), which depends here on the outer iteration n. This iteration dependent tolerance seems reasonable to us, since we realized in our numerical tests, that in the first outer iterations the subdomain problems do not need to be solved very accurately, due to the averaging of the current and previous iterates in the update of u^n, see Algorithm 2.5. Therefore we set

$$\text{tol}(n) = \begin{cases} 10^{-4} & \text{if } 1 \le n \le 4 \\ 5 \cdot 10^{-5} & \text{if } 4 < n \le 7 \\ 10^{-5} & \text{if } 7 < n \le 50 \\ 8 \cdot 10^{-6} & \text{if } 50 < n \le 400 \\ 5 \cdot 10^{-6} & \text{if } n > 400 \end{cases},$$

which is chosen empirically and not optimized in any way. We consider partitions of the image domain into $M = 2, 4, 8$ subdomains and utilize Algorithm 2.5 to compute a solution of (1.1). For a splitting into 4 and 8 domains we consider a decomposition without and with colouring technique. In case of using the colouring technique the domains are coloured as described in Figure 5 leading to $M_c = 2$. Since we are comparing the convergence speed of different algorithms, we terminate the algorithms as soon as the energy J drops below a certain critical energy J^* for the first time. This critical energy is obtained empirically by solving the global problem very accurately, so that J^* is very close to the true minimum.

For our comparison we consider the image in Figure 10(a) of size 1024×1024 pixels, which has been corrupted by additive Gaussian noise with standard deviation $\sigma = 0.1$ and zero mean. In the parallel domain decomposition algorithm as well as in the split Bregman algorithm we denoise the image by choosing $\alpha = 1$. In the parallel domain decomposition algorithm the size of the overlapping region is set to be 1024×20 pixels. In Table 1 we show for different numbers of subdomains the required time (in seconds) and the number of iterations until the algorithms reached the significant energy $J^* = 0.018178900879$. The restored image is shown in Figure 10(b). Note, that by domain decomposition, on the one hand, we reduce the dimensionality of the problem, but, on the other hand, in each outer iteration the update v_i^{n+1} is a weighted sum of the current and previous iterate. This averaging is needed for theoretical reasons, in particular to ensure the convergence to the minimizer of the global problem. The colouring technique reduces the averaging effect, which leads to a faster convergence. It is obvious that with increasing number of subdomains M this effect becomes more and more visible, see Table 1. Additionally, we also have to take the communication time of the processors into account. All these facts sum up to the actual computing time. Hence, we cannot expect a very dramatic decrease in computational time. Nevertheless, we observe from Table 1 that the domain decomposition algorithm with splitting into $M = 2, 4, 8$ subdomains is faster than the split Bregman algrorithm computing the solution on the whole domain (1 domain). Thereby, for a decomposition into 8 domains using the colouring technique the best performance with respect to time and iterations is obtained. More precisely, using the parallel domain decomposition method with 8 domains and colouring technique reduces the overall computational time by more than 40% compared with no decomposition.
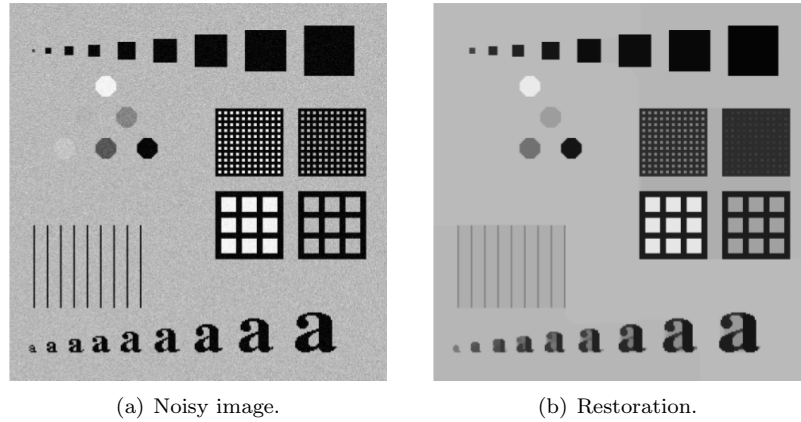

(a) Noisy image.   (b) Restoration.

FIG. 10. *Reconstruction of an image of size* 1024×1024 *pixels corrupted by additive Gaussian white noise with* $\sigma = 0.1$.

5. Conclusion. We developed convergent overlapping domain decomposition methods for the Rudin-Osher-Fatemi (ROF) problem (1.1) by directly splitting the (primal) problem into respective subdomain problems. In particular, we proved convergence of our proposed splitting methods to a minimizer of the global problem in a continuous setting. We presented two different ways to solve the subdomain problems leading to two similar but still different implementations.

TABLE 1
Regularization parameter $\alpha = 1$, stopping criterion $J^* = 0.018178900879$.

	CPU time (s)	No. outer iterations
1 domain	34943.73	9738
2 domains	29228.35	4571
4 domains	22708.67	4470
4 domains with colouring	22205.89	4451
8 domains	21156.57	4253
8 domains with colouring	20481.00	4147

Due to the shape of our subdomain problems, the presented domain decomposition methods are easily applicable to optimization problems with a spatially varying regularization weight, i.e. to problems of the form

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|u - g\|_{L^2(\Omega)}^2 + \int_\Omega \alpha |Du|$$

where $\alpha : \Omega \to \mathbb{R}^+$ is a continuous and bounded function, cf. [39, 41]. Recently, this type of problem is gaining more and more attention, since it allows to penalize homogeneous regions strongly, while in image parts with fine details only little regularization is performed.

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Appendix A. Solving (3.4) with respect to $u_{1,\Omega_1^h}^{h,k+1}$.

In order to get the solution $u_{1,\Omega_1^h}^{h,k+1}$ in (3.4), we only need to solve a linear system. We describe here how this linear system looks when the domain is split vertically into overlapping stripes, as in Section 4.1. Note, that only the size of the resulting linear problem depends on the number of subdomains M, but not its structure.

Let the image domain $\Omega^h = \{x_1^1 < x_2^1 < \cdots < x_{N_1}^1\} \times \{x_1^2 < x_2^2 < \cdots < x_{N_2}^2\}$ be a rectangular domain consisting of $N_1 \times N_2$ discrete pixels. Then we decompose Ω^h into overlapping subdomains Ω_i^h, $i = 1, \ldots, M$, such that $\Omega_1^h = \{x_1^1 < \cdots < x_L^1\} \times \{x_1^2 < \cdots < x_{N_2}^2\}$ and $\Omega^h \backslash \Omega_1^h = \{x_{L+1}^1 < \cdots < x_{N_1}^1\} \times \{x_1^2 < \cdots < x_{N_2}^2\}$. Further we set $\tilde{\Omega}_1^h = \{x_{L+1}^1\} \times \{x_1^2 < \cdots < x_{N_2}^2\}$. We define a restriction of the global operator div_Ω^h to the domain Ω_1^h by the local discrete divergent operator $\tilde{\text{div}}_{\Omega_1}^h$ as

$$(\tilde{\text{div}}_{\Omega_1}^h \mathbf{p}^h)(x_i^1, x_j^2) = \begin{cases} p^{h,1}(x_i^1, x_j^2) & \text{if } i = 1 \\ p^{h,1}(x_i^1, x_j^2) - p^{h,1}(x_{i-1}^1, x_j^2) & \text{if } 1 < i \leq L \end{cases}$$
$$+ \begin{cases} p^{h,2}(x_i^1, x_j^2) & \text{if } j = 1 \\ p^{h,2}(x_i^1, x_j^2) - p^{h,2}(x_i^1, x_{j-1}^2) & \text{if } 1 < j < N_2 \\ -p^{h,2}(x_i^1, x_{j-1}^2) & \text{if } j = N_2 \end{cases}$$

for every $\mathbf{p}^h = (p^{h,1}, p^{h,2}) \in Y_1$. Accordingly, we denote the associated Laplace operator by $\tilde{\Delta}_{\Omega_1}^h$ defined as

$$\tilde{\Delta}_{\Omega_1}^h u^h(x_i^1, x_j^2) = \begin{cases} u^h(x_i^1, x_j^2) - u^h(x_{i+1}^1, x_j^2) & \text{if } i = 1 \\ 2u^h(x_i^1, x_j^2) - u^h(x_{i+1}^1, x_j^2) - u^h(x_{i-1}^1, x_j^2) & \text{if } 1 < i \leq L. \end{cases}$$
$$+ \begin{cases} u^h(x_i^1, x_j^2) - u^h(x_i^1, x_{j+1}^2) & \text{if } j = 1 \\ 2u^h(x_i^1, x_j^2) - u^h(x_i^1, x_{j-1}^2) - u^h(x_i^1, x_{j+1}^2) & \text{if } 1 < j < N_2 \\ u^h(x_i^1, x_j^2) - u^h(x_i^1, x_{j-1}^2) & \text{if } j = N_2 \end{cases}$$

With the above notations the optimality of $u_{1,\Omega_1^h}^{h,k+1}$ in (3.4) is equivalent to the solution $u_{1|_{\Omega_1^h}}^h$ of the following boundary value problem

$$(A.1) \qquad u_1^h(x) - \mu \tilde{\Delta}_{\Omega_1}^h u_1^h(x) = f_1^{h,n+1}(x) + \tilde{\mathrm{div}}_{\Omega_1}^h (\mathbf{b}_1^k - \mathbf{d}_1^h)(x), \qquad x \in \Omega_1^h$$

$$(A.2) \qquad u_1^h(x) = f_1^{h,n+1}(x), \qquad x \in \tilde{\Omega}_1^h.$$

The system (A.1)-(A.2) may be also written as a linear system

$$A^h v^h = b^h,$$

where the vector $v^h \in \mathbb{R}^{N_1 L}$ has the values $u_{1|_{\Omega_1^h}}^h$ as its components arranged in a particular order, $A^h \in \mathbb{R}^{N_1 L \times N_1 L}$ and $b^h \in \mathbb{R}^{N_1 L}$ mimicking the associated matrix constituted from the left-hand side and the right-hand side of (A.1) together with the boundary conditions (A.2), respectively.

We remark, that similar considerations yield the linear system for obtaining $u_{i,\Omega_i^h}^{h,k+1}$, $i = 2,\dots,M$.

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